IB MATH AA HL ▪ VECTORS

The complete notes

AHL 3.12–3.16 ▪ Photon Academy ▪ ibtuition.sg

How to use this book

Vectors is one strand, not 35 unrelated topics, and this book is arranged the way the strand actually builds.

20 basics come first in each session. They are the trunk: small, single-step skills you should be able to do without thinking, because every exam question is made of them. Each one gets the idea, an examiner's note, and two worked examples.

15 exam-question types follow, 61 parts in total. These are the real thing — a Section B question gives you one configuration and then asks (a), (b), (c)..., each part leaning on the one before. Each part gets its own idea, worked solution, and the mistakes examiners see most.

Everything here is generated from Photon's own Learning Studio question bank, so the worked examples are the same ones the app sets — with fresh numbers every time you open it.

Contents

1 Session 1 — Vectors I — notation and algebra

AHL 3.12 • 4 basics • 4 exam-question parts

1.1 Notation

A vector wears two dresses. **Column form** stacks the components; **base-vector form** writes them against $\mathbf{i}, \mathbf{j}, \mathbf{k}$. A zero component means that base vector is simply absent — it is not written.

Examiner Insight

Column form is quicker for subtracting position vectors, and a sign slip is easier to spot in it.

Worked Example

Write $\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ in column form.

Working. $\mathbf{i} - 3\mathbf{j} + 6\mathbf{k} = \begin{pmatrix} 1 \\ -3 \\ 6 \end{pmatrix}$ — a missing base vector is a zero in that row.

Worked Example

Write $2\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$ in column form.

Working. $2\mathbf{i} - 3\mathbf{j} - 3\mathbf{k} = \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix}$ — a missing base vector is a zero in that row.

1.2 Adding and scaling

Scaling multiplies **every** component; adding and subtracting work row by row. There is no cross-talk between rows.

Examiner Insight

Scale first and write the two intermediate columns down — most slips here are arithmetic, not method.

Worked Example

Given $\mathbf{u} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$, find $4\mathbf{u} - 3\mathbf{v}$.

Working. $\begin{pmatrix} -4 \\ -8 \\ 4 \end{pmatrix} - \begin{pmatrix} 9 \\ 9 \\ 3 \end{pmatrix} = \begin{pmatrix} -13 \\ -17 \\ 1 \end{pmatrix}$.

Worked Example

Given $\mathbf{u} = \begin{pmatrix} 3 \\ 4 \\ -3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}$, find $2\mathbf{u} - 2\mathbf{v}$.

Working. $\begin{pmatrix}6\\8\\-6\end{pmatrix} - \begin{pmatrix}4\\-8\\-6\end{pmatrix} = \begin{pmatrix}2\\16\\0\end{pmatrix}$.

1.3 AB = b − a

The most-used line in the whole strand. The displacement from A to B is **head minus tail**:

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}.$$

Writing $\mathbf{a} - \mathbf{b}$ gives $\overrightarrow{BA}$ — same length, every sign flipped.

Examiner Insight

Say it as you write it: 'to B, from A — so B minus A.'

Worked Example

$A(-4,\ -5,\ -4)$ and $B(-1,\ -1,\ 1)$. Find $\overrightarrow{AB}$.

Working. $\overrightarrow{AB} = \begin{pmatrix}-1\\-1\\1\end{pmatrix} - \begin{pmatrix}-4\\-5\\-4\end{pmatrix} = \begin{pmatrix}3\\4\\5\end{pmatrix}$.

Worked Example

$A(0,\ 6,\ -2)$ and $B(-5,\ 5,\ -5)$. Find $\overrightarrow{AB}$.

Working. $\overrightarrow{AB} = \begin{pmatrix}-5\\5\\-5\end{pmatrix} - \begin{pmatrix}0\\6\\-2\end{pmatrix} = \begin{pmatrix}-5\\-1\\-3\end{pmatrix}$.

1.4 The midpoint

The midpoint of A and B is the **average of the position vectors**:

$$\mathbf{m} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b}).$$

It is halfway along $\overrightarrow{AB}$, so it is also $\mathbf{a} + \frac{1}{2}\overrightarrow{AB}$ — the same point reached two ways.

Examiner Insight

Averaging is the fast route. If a question gives you the midpoint and one end, rearrange: $\mathbf{b} = 2\mathbf{m} - \mathbf{a}$ — the same move as reflecting a point.

Worked Example

$M(0,\ 1,\ 3)$ is the midpoint of PQ, and $P(0,\ 2,\ 1)$. Find Q.

Working. $\mathbf{q} = 2\begin{pmatrix}0\\1\\3\end{pmatrix} - \begin{pmatrix}0\\2\\1\end{pmatrix} = \begin{pmatrix}0\\0\\5\end{pmatrix}$.

Worked Example

Find the midpoint of $A(6,\ -4,\ 6)$ and $B(6,\ -5,\ -2)$.

Working. $\frac{1}{2}\left(\begin{pmatrix}6\\-4\\6\end{pmatrix}+\begin{pmatrix}6\\-5\\-2\end{pmatrix}\right)=\begin{pmatrix}6\\-4.5\\2\end{pmatrix}$.

1.5 Geometry without coordinates

A figure, two base vectors, and not a single coordinate. How the IB tests whether you understand vectors rather than arithmetic.

The shape of the question

$OACB$ is a parallelogram with $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$. The point P lies on AB.

(a) The diagonal

Reading a route through a figure is just addition: $\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}$. In the parallelogram $\overrightarrow{AC}$ **equals** $\overrightarrow{OB}=\mathbf{b}$ — equal vectors, drawn somewhere else. So the diagonal from the shared corner is the **sum**.

Examiner Insight

Equal vectors are the whole trick: same length, same direction, position irrelevant.

Worked — part (a)

Express $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Working. $\overrightarrow{OC}=\mathbf{a}+\mathbf{b}$ — the parallelogram law.

Answer. $\begin{pmatrix}1\\1\end{pmatrix}$

(b) The other diagonal

One diagonal is the sum; the other is the **difference**: $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$. The same head-minus-tail rule from the roots, now with no coordinates in sight.

Examiner Insight

Sum and difference give the two diagonals. Worth remembering as a pair.

Worked — part (b)

Express $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Working. $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$, i.e. $(-1)\mathbf{a}+(1)\mathbf{b}$.

Answer. $\begin{pmatrix}-1\\1\end{pmatrix}$

Common Pitfall

You wrote $\mathbf{a} - \mathbf{b}$, which is $\overrightarrow{BA}$. From A to B is $\mathbf{b} - \mathbf{a}$.

(c) A point part-way along

You have $\overrightarrow{AB}$. Travel a fraction λ of it from A:

$$\overrightarrow{OP} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}.$$

The weights always sum to 1 — the signature of a point *on the segment*.

Examiner Insight

Check every answer of this kind: do the coefficients add to 1?

Worked — part (c)

P lies on AB with $\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AB}$. Express $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Working. $\overrightarrow{OP} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$. Weights sum to 1 ✓

Answer. $\begin{pmatrix} 0.500 \\ 0.500 \end{pmatrix}$

Common Pitfall

Right pair, wrong way round. The coefficient of $\mathbf{a}$ is $1 - \lambda$ — you start *at* A and travel λ away from it.

(d) Given as a ratio instead

A ratio $AP : PB = m : n$ just names $\lambda = \frac{m}{m+n}$, so

$$\overrightarrow{OP} = \frac{n}{m+n}\mathbf{a} + \frac{m}{m+n}\mathbf{b}.$$

Note the swap — the weight on $\mathbf{a}$ carries the *far* part of the ratio, because the further you travel from A, the less of $\mathbf{a}$ remains.

Examiner Insight

Sanity-check with an extreme: if P were at A the ratio is $0 : n$, and the formula returns $\mathbf{a}$ ✓

Worked — part (d)

Instead, P divides AB with $AP : PB = 3 : 2$. Express $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Working. $\overrightarrow{OP} = \frac{2}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}$.

Answer. $\begin{pmatrix} 0.400 \\ 0.600 \end{pmatrix}$

Common Pitfall

You put $\frac{3}{5}$ on $\mathbf{a}$. The ratio is measured from A, so travelling further from A *reduces* the weight on $\mathbf{a}$.

2 Session 2 — Vectors II — magnitude and direction

AHL 3.12 • 4 basics • 4 exam-question parts

2.1 Magnitude

Pythagoras, in as many dimensions as you have:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Never negative, and squaring removes every sign — so $|\overrightarrow{AB}| = |\overrightarrow{BA}|$.

Examiner Insight

On Paper 1 leave it as a surd. $\sqrt{29}$ is a finished answer; 5.39 is not.

Worked Example

Find $\left|\begin{pmatrix}0\\3\\4\end{pmatrix}\right|$.

Working. $\sqrt{(0)^2 + (3)^2 + (4)^2} = \sqrt{25} = 5.$

Worked Example

Find $\left|\begin{pmatrix}2\\-6\\9\end{pmatrix}\right|$.

Working. $\sqrt{(2)^2 + (-6)^2 + (9)^2} = \sqrt{121} = 11.$

2.2 Unit vector

Dividing by the magnitude strips out the length and keeps only the direction:

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

Check it — the components of a unit vector square-and-sum to exactly 1.

Examiner Insight

Write it as one fraction times a column, $\frac{1}{3}\begin{pmatrix}2\\-1\\2\end{pmatrix}$ — the mark scheme's preferred form.

Worked Example

Find the unit vector in the direction of $\begin{pmatrix}-6\\6\\7\end{pmatrix}$.

Working. $\hat{\mathbf{a}} = \frac{1}{11}\begin{pmatrix}-6\\6\\7\end{pmatrix} = \begin{pmatrix}-0.545\\0.545\\0.636\end{pmatrix}$.

Worked Example

Find the unit vector in the direction of $\begin{pmatrix} 4 \\ 4 \\ -7 \end{pmatrix}$.

Working. $\hat{\mathbf{a}} = \frac{1}{9}\begin{pmatrix} 4 \\ 4 \\ -7 \end{pmatrix} = \begin{pmatrix} 0.444 \\ 0.444 \\ -0.778 \end{pmatrix}$.

2.3 Parallel vectors

Parallel means one is a scalar multiple of the other, $\mathbf{a} = k\mathbf{b}$ — and the **same** k must work in every row. Matching one row and stopping is the commonest error in this sub-topic.

Examiner Insight

Get k from a row with no unknown in it, then use it on the row that has one.

Worked Example

$\begin{pmatrix} 2 \\ t \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$ are parallel. Find t.

Working. $k = -0.5$ (check row 3: $2 \times -0.5 = -1$ ✓), so $t = -1 \times -0.5 = 0.5$.

Worked Example

$\begin{pmatrix} 1.5 \\ t \\ -0.5 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$ are parallel. Find t.

Working. $k = -0.5$ (check row 3: $1 \times -0.5 = -0.5$ ✓), so $t = -1 \times -0.5 = 0.5$.

2.4 Parallel and anti-parallel

$\mathbf{v} = k\mathbf{u}$ makes the two **parallel** whatever k is. The sign of k says which way: $k > 0$ and they point the **same way**; $k < 0$ and they point **opposite ways** — *anti-parallel*, still parallel as lines. If no single k works, they are not parallel at all.

Examiner Insight

A direction vector and its negative describe the same line. That is why an answer 'up to scale' accepts $-\mathbf{d}$ as readily as $\mathbf{d}$.

Worked Example

How do $\begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -6 \\ 4 \end{pmatrix}$ sit relative to one another?

A. Parallel, same direction

B. Anti-parallel

C. Not parallel

Working. $\mathbf{v} = 2\,\mathbf{u}$, and the multiplier is positive, so they point the same way.

Worked Example

How do $\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 12 \\ -6 \end{pmatrix}$ sit relative to one another?

A. Parallel, same direction

B. Anti-parallel

C. Not parallel

Working. $\mathbf{v} = 3\,\mathbf{u}$, and the multiplier is positive, so they point the same way.

2.5 Three points

Given three points, decide how they sit relative to one another — the standard opening of a Section A vectors question.

The shape of the question

The points $A(2,\ -5,\ 2)$, $B(5,\ -5,\ 6)$ and $C(14,\ -5,\ 18)$ are given.

(a) The displacement

Start where every one of these questions starts: $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$. The rest of the question is built on this one column.

Examiner Insight

Write it down clearly — you will be reading it back three more times.

Worked — part (a)

Find $\overrightarrow{AB}$.

Working. $\overrightarrow{AB} = \begin{pmatrix} 5 \\ -5 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$.

Answer. $\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$

Common Pitfall

That is $\overrightarrow{BA}$. Head minus tail: $\mathbf{b} - \mathbf{a}$.

(b) Hence the distance

Distance is the magnitude of the displacement you already have: $AB = |\overrightarrow{AB}|$. Two steps in a fixed order — **subtract, then measure**.

Examiner Insight

If you get a negative under the root, you subtracted after squaring. Subtract first.

Worked — part (b)

Hence find the distance AB.

Working. $AB = \sqrt{25} = 5$.

Answer. 5

(c) The midpoint

A midpoint is an **average of position vectors**, not a difference:

$$\overrightarrow{OM} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b}).$$

Compare with $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ from part (a). One names a place; the other names a journey.

Examiner Insight

Ask which the question wants: a point in space, or a movement between two points?

Worked — part (c)

Find the position vector of the midpoint M of AB.

Working. $\overrightarrow{OM} = \frac{1}{2}\left(\begin{pmatrix} 2 \\ -5 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ -5 \\ 6 \end{pmatrix}\right) = \begin{pmatrix} 3.5 \\ -5 \\ 4 \end{pmatrix}$.

Answer. $\begin{pmatrix} 3.50 \\ -5 \\ 4 \end{pmatrix}$

Common Pitfall

That is $\overrightarrow{AB}$ again — the displacement. A midpoint is the *average*.

(d) Show they are collinear

Three points are collinear when two displacements **from a shared point** are parallel: $\overrightarrow{AC} = k\,\overrightarrow{AB}$. The shared point is what makes it one line rather than two parallel ones — and that k is the ratio.

Examiner Insight

The same k must work in every row. This is the parallel test from the roots, used as a proof.

Worked — part (d)

Show that A, B and C are collinear by finding k with $\overrightarrow{AC} = k\,\overrightarrow{AB}$.

Working. $\overrightarrow{AC} = \begin{pmatrix} 12 \\ 0 \\ 16 \end{pmatrix} = 4\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$, so $k = 4$ and $AB : AC = 1 : 4$.

Answer. 4

3 Session 3 — The scalar (dot) product

AHL 3.13 • 4 basics

3.1 The scalar product

Multiply matching components and add:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

The result is a **number**, not a vector — that is what 'scalar' means.

Examiner Insight

Keep the signs. A negative dot product is a real answer, and it tells you the angle is obtuse.

Worked Example

Find $\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 4 \\ -1 \end{pmatrix}$.

Working. $(4)(-5) + (2)(4) + (0)(-1) = -12$.

Worked Example

Find $\begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -3 \\ -5 \end{pmatrix}$.

Working. $(3)(5) + (4)(-3) + (4)(-5) = -17$.

3.2 What the sign of the dot product tells you

The dot product carries the angle in its **sign**, before you compute anything: $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$, and the magnitudes are never negative. So **positive means acute**, **zero means perpendicular**, **negative means obtuse**. One arithmetic line answers 'is this angle bigger or smaller than $90°$'.

Examiner Insight

Examiners ask this without the word 'dot product' — 'show the angle is obtuse' means show the dot product is negative, and nothing more.

Worked Example

Without finding the angle, decide whether the angle between $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$ is acute, right or obtuse.

A. Acute

B. A right angle

C. Obtuse

Working. $\mathbf{u} \cdot \mathbf{v} = -11$, which is **negative**, so the angle is obtuse.

Worked Example

Without finding the angle, decide whether the angle between $\begin{pmatrix}-4\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}-4\\-3\\-3\end{pmatrix}$ is acute, right or obtuse.

A. Acute
B. A right angle
C. Obtuse

Working. $\mathbf{u}\cdot\mathbf{v} = 16$, which is **positive**, so the angle is acute.

3.3 Angle between vectors

The two forms of the dot product together give the angle:

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

Place them **tail to tail**, so $0^\circ \le \theta \le 180^\circ$.

Examiner Insight

On Paper 1 stop at the exact cosine; on Paper 2 give the angle to 3 s.f.

Worked Example

Find the angle between $\begin{pmatrix}-1\\-2\\2\end{pmatrix}$ and $\begin{pmatrix}-2\\-1\\-2\end{pmatrix}$, in degrees to 1 d.p.

Working. $\cos\theta = 0$, so $\theta = 90^\circ$.

Worked Example

Find the angle between $\begin{pmatrix}-6\\-6\\-7\end{pmatrix}$ and $\begin{pmatrix}-6\\2\\-3\end{pmatrix}$, in degrees to 1 d.p.

Working. $\cos\theta = 0.584$, so $\theta = 54.238^\circ$.

3.4 Perpendicular

The cleanest condition in the strand:

$$\mathbf{a}\perp\mathbf{b} \iff \mathbf{a}\cdot\mathbf{b} = 0.$$

Set the dot product to zero and solve — almost always one linear equation.

Examiner Insight

Substitute your answer back and check the dot product really is zero. Five seconds, one mark.

Worked Example

$\begin{pmatrix}-1\\t\\1\end{pmatrix}$ and $\begin{pmatrix}2\\-1\\-4\end{pmatrix}$ are perpendicular. Find t.

Working. $-6 - t = 0 \Rightarrow t = -6$.

Worked Example

$\begin{pmatrix}2\\t\\2\end{pmatrix}$ and $\begin{pmatrix}-3\\2\\-3\end{pmatrix}$ are perpendicular. Find t.

Working. $-12 + 2t = 0 \Rightarrow t = 6$.

4 Session 4 — Geometric vector proofs

AHL 3.12–3.13 • 8 exam-question parts

4.1 Find the value of k

A configuration with a letter left in it. Each condition — perpendicular, collinear, a given area — turns into one equation in k, and the papers set this more often than any other opening.

The shape of the question

The points $A(-3,\ 0,\ -3)$, $B(k,\ -9,\ 6)$ and $C(-1,\ 3,\ -6)$ are given, where $k \in \mathbb{R}$.

(a) Perpendicular

Write $\overrightarrow{AB}$ with the k still in it — only one component carries it — and $\overrightarrow{AC}$ as usual. Perpendicular means

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = 0,$$

and because k appears once and linearly, that dot product is a **linear equation in** k. One line of algebra.

Examiner Insight

Do not substitute a number for k first and hope. Carry the letter through the dot product and solve at the end — that is the whole method, and it is worth full marks.

Worked — part (a)

Find the value of k for which $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$.

Working. $(k+3)(2) + (-9)(3) + (9)(-3) = 0$ gives $k = 24$.

Answer. 24

Common Pitfall

That makes only the **first** component of $\overrightarrow{AB}$ zero. Perpendicular needs the whole dot product to vanish, so the other two terms have to be balanced too.

Common Pitfall

That is the value that makes the points **collinear** — the opposite of perpendicular. Check which condition you were given.

(b) Collinear

Collinear means $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are **parallel**: $\overrightarrow{AB} = t\,\overrightarrow{AC}$ for some t. The two components that carry no k fix t straight away; the remaining component then gives k.

Examiner Insight

They share the point A, so parallel is enough — you do not have to show anything else. Two vectors that are parallel *and* share a point are on one line.

Worked — part (b)

Find the value of k for which A, B and C are collinear.

Working. From the y and z components, $t = -3$. Then $k + 3 = -3 \times 2$, so $k = -9$.

Answer. -9

Common Pitfall

That is the **perpendicular** value from part (a). Collinear is the parallel condition, not the dot-product one.

(c) Hence the line through all three

With that k the three points are on one line, so any of them serves as the point and $\overrightarrow{AC}$ as the direction. Nothing new to compute — the parts before it have already produced both pieces.

Examiner Insight

Any of A, B or C is an acceptable point, and any non-zero multiple of the direction is acceptable too. Examiners mark the line, not the particular way you wrote it.

Worked — part (c)

Taking that value of k, write down a vector equation for the line through A, B and C.

Working. $\mathbf{r} = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}$ — and B sits on it at $\lambda = -3$.

Answer. $\mathbf{r} = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}$

(d) And with a different k, the area

Change k and the points stop being collinear, so they now make a genuine triangle. Its area is half the cross product of two sides from the same vertex:

$$\tfrac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|.$$

Examiner Insight

If this ever comes out zero, the points were collinear after all — which is a neat way to check part (b) rather than a failure.

Worked — part (d)

Now let $k = -4$. Find the area of triangle ABC, to 3 s.f.

Working. $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 0 \\ 15 \\ 15 \end{pmatrix}$, so the area is $\frac{1}{2}\,|\cdot| = 10.607$.

Answer. 10.6

Common Pitfall

That is the **parallelogram**. A triangle is half of it — the $\frac{1}{2}$ is not optional.

4.2 Show that it is always true

A Section B proof. No coordinates to grind — the marks are for the argument, and each part is one algebraic step you have to justify rather than compute.

The shape of the question

For any two vectors $\mathbf{u}$ and $\mathbf{v}$, with θ the angle between them, you are asked to prove

$$|\mathbf{u} \times \mathbf{v}|^2 + (\mathbf{u} \cdot \mathbf{v})^2 = |\mathbf{u}|^2\,|\mathbf{v}|^2\,.$$

(a) Write the dot product with the angle in it

Both sides of the identity are about the **angle**, so start by putting the angle into each piece. The dot product's definition does that immediately: $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$, so squaring gives $|\mathbf{u}|^2|\mathbf{v}|^2\cos^2\theta$.

Examiner Insight

In a 'prove' question, quoting the definition **is** a marked step. Write it down rather than using it silently.

Worked — part (a)

Check the first piece numerically with $\mathbf{u} = \begin{pmatrix} -4 \\ -1 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ -4 \\ 4 \end{pmatrix}$: find $(\mathbf{u} \cdot \mathbf{v})^2$.

Working. $\mathbf{u} \cdot \mathbf{v} = -12$, so $(\mathbf{u} \cdot \mathbf{v})^2 = 144$.

Answer. 144

Common Pitfall

That is $\mathbf{u} \cdot \mathbf{v}$ itself. The identity uses its **square**.

(b) And the cross product with the angle in it

The cross product carries the **sine** of the same angle: $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$. Squaring gives $|\mathbf{u}|^2|\mathbf{v}|^2 \sin^2\theta$ — the sine partner to the piece you just wrote.

Examiner Insight

This is why the two products belong together. One measures how much the vectors agree, the other how much they differ, and the angle is the same in both.

Worked — part (b)

Now find $|\mathbf{u} \times \mathbf{v}|^2$ for the same two vectors.

Working. $\left|\begin{pmatrix}-8\\13\\19\end{pmatrix}\right|^2 = 594.$

Answer. 594

Common Pitfall

That is $|\mathbf{u} \times \mathbf{v}|$. The identity needs it **squared**, which saves you taking the square root at all.

(c) Add them and use the Pythagorean identity

Adding the two gives

$$|\mathbf{u}|^2|\mathbf{v}|^2 \left(\sin^2\theta + \cos^2\theta\right),$$

and the bracket is 1. The angle disappears, leaving $|\mathbf{u}|^2|\mathbf{v}|^2$ — which is the result. That single Pythagorean step is the whole proof.

Examiner Insight

State where $\sin^2\theta + \cos^2\theta = 1$ is used. A proof that jumps from the sum to the answer without naming it loses the reasoning mark.

Worked — part (c)

Add your two answers together. What do you get?

Working. $594 + 144 = 738$, and $|\mathbf{u}|^2|\mathbf{v}|^2 = 18 \times 41 = 738$ ✓

Answer. 738

(d) Confirm the right-hand side

The numbers agreeing does not prove anything on its own — one example never does. But it confirms you have not slipped, and the **argument** in the parts above, written with θ rather than numbers, is what proves it for every pair of vectors.

Examiner Insight

Know the difference: a check verifies one case, a proof covers all of them. An exam asking you to 'show that' wants the algebra, not an example.

Worked — part (d)

Finally, compute $|\mathbf{u}|^2\,|\mathbf{v}|^2$ directly.

Working. $18 \times 41 = 738$ — equal to the sum from part (c), as the identity says it must be.

Answer. 738

5 Session 5 — The vector equation of a line

AHL 3.14 • 2 basics • 4 exam-question parts

5.1 The forms of a line

A line has three faces and they all say the same thing. **Vector**: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$. **Parametric**: the three rows written out, $x = a_1 + \lambda d_1$ and so on. **Cartesian**: eliminate λ from those rows to get

$$\frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}.$$

Reading a Cartesian form backwards is the same move in reverse: the numbers subtracted from x, y, z are the **point**, the denominators are the **direction**.

Examiner Insight

The commonest slip is a sign: $\dfrac{x+4}{3}$ means the point has $x = -4$, not $+4$. Cartesian form subtracts the coordinate.

Worked Example

In the plane, a line passes through $(-5,\ -1)$ and $(5,\ 6)$. Write down a vector equation for it.

Working. $\overrightarrow{PQ} = \begin{pmatrix} 10 \\ 7 \end{pmatrix}$, so $\mathbf{r} = \begin{pmatrix} -5 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 10 \\ 7 \end{pmatrix}$.

Worked Example

A line passes through $(-1,\ -3,\ 5)$ with direction $\begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$. Write down its vector equation.

Working. $\mathbf{r} = \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$. Its Cartesian form would be

$$\frac{x+1}{-2} = \frac{y+3}{3} = \frac{z-5}{2}.$$

5.2 What the parameter does

In $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, λ **is a dial**. Every value gives one point of the line, and every point of the line has exactly one λ. $\lambda = 0$ sits on $\mathbf{a}$; $\lambda = 1$ is one step of $\mathbf{d}$ further on; negative λ goes back the other way.

Examiner Insight

To test whether a point lies on a line, solve for λ in the first row — then check that **the same** λ works in the other two. One row agreeing proves nothing.

Worked Example

A line is $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. The point $(-1,\ 1,\ -4)$ lies on it. Find λ.

Working. The first row gives $\lambda = -2$, and it satisfies all three rows ✓

Worked Example

A line is $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}$. The point $(-8,\ 9,\ 3)$ lies on it. Find λ.

Working. The first row gives $\lambda = 3$, and it satisfies all three rows ✓

5.3 The equation of a line

Turn two points into a line, move between its forms, and test whether a point lies on it.

The shape of the question

The line L passes through $A(0,\ -3,\ 3)$ and $B(2,\ -5,\ 0)$. The point $P(2,\ -5,\ 0)$ is also given.

(a) A direction vector

A line needs **a point on it** and **a direction along it**. Through A and B the direction is $\overrightarrow{AB}$ — the root skill again, doing a new job.

Examiner Insight

Any non-zero multiple of the direction describes the same line, so simplify it if you can.

Worked — part (a)

Find a direction vector for L.

Working. $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}$. Any non-zero multiple is equally valid.

Answer. $\begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}$

(b) Hence the vector equation

Point plus direction is the whole equation:

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}.$$

Either given point works and any parallel direction works — one line has infinitely many correct equations, and the mark scheme accepts all of them.

Examiner Insight

Answer with whatever point and direction you actually computed. An equivalent form is not penalised.

Worked — part (b)

Write down a vector equation for L.

Working. $\mathbf{r} = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}$. Using B, or a multiple of the direction, is equally correct.

Answer. $\mathbf{r} = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}$

(c) In Cartesian form

Cartesian form is the parameter eliminated:

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}.$$

Numerators give a point, denominators give the direction. Watch the signs — $\frac{x+3}{2}$ means $x_0 = -3$.

Examiner Insight

Converting back is pure reading. No algebra at all.

Worked — part (c)

The line is also written $\frac{x-0}{2} = \frac{y+3}{-2} = \frac{z-3}{-3}$. Convert it back to vector form.

Working. $\mathbf{r} = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}$ — the line you already had.

Answer. $\mathbf{r} = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}$

(d) Does P lie on L?

A point lies on the line only if **one single** λ satisfies all three rows. Find λ from one row, then it must check out in the other two. One row agreeing proves nothing.

Examiner Insight

The same discipline as 'verify with the third equation' when two lines might intersect.

Worked — part (d)

Determine whether $P(2,\ -5,\ 0)$ lies on L.

A. Yes — one value of λ satisfies all three rows

B. No — the rows demand different values of λ

Working. Row 1 gives $\lambda = 1$, which also satisfies rows 2 and 3. P is on L.
Answer. **A** — Yes — one value of λ satisfies all three rows

6 Session 6 — Relationships between lines

AHL 3.14–3.15 • 1 basic • 3 exam-question parts

6.1 How two lines can sit

Two lines in space do one of **four** things. **Coincident** — same line, written twice: the directions are multiples *and* a point of one lies on the other. **Parallel and distinct** — directions are multiples, but no shared point. **Intersecting** — not parallel, and one pair of λ, μ satisfies *all three* rows. **Skew** — not parallel, yet no pair works: they pass without meeting, which cannot happen in two dimensions.

Examiner Insight

Parallel directions are not enough to call two lines parallel — test whether a point of one lies on the other, or you will call a coincident pair 'parallel'. And skew is a *proved* result, not a shrug: solve two rows, substitute into the third, show it fails.

Worked Example

$L_1 : \ \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -1 \\ -1 \end{pmatrix}$ and $L_2 : \ \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix}$. How do they sit?

A. Coincident — the same line

B. Parallel and distinct

C. Intersecting

D. Skew

Working. $\mathbf{d}_2$ is a multiple of $\mathbf{d}_1$ but $\mathbf{a}_2$ is not on L_1, so they are **parallel and distinct**.

Worked Example

$L_1 : \ \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ and $L_2 : \ \mathbf{r} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ -2 \\ 6 \end{pmatrix}$. How do they sit?

A. Coincident — the same line

B. Parallel and distinct

C. Intersecting

D. Skew

Working. $\mathbf{d}_2$ is a multiple of $\mathbf{d}_1$ **and** $\mathbf{a}_2$ lies on L_1 — the two equations describe the **same line**.

6.2 Two lines in space

Classify a pair of lines, find where they meet, and measure the angle between them. The densest-examined type in the strand.

The shape of the question

$$L_1 : \mathbf{r} = \begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \text{ and } L_2 : \mathbf{r} = \begin{pmatrix} 2 \\ -5 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

(a) How are they related?

Three questions, in order. **Are the directions parallel?** If yes, the lines are parallel. If no, solve two of the three row-equations for λ and μ, then **substitute into the third**. Holds $\Rightarrow$ intersecting. Fails $\Rightarrow$ skew.

Examiner Insight

Skew is not 'I could not solve it'. It is what you have *proved* when two rows agree and the third does not.

Worked — part (a)

Determine whether L_1 and L_2 are parallel, intersecting or skew.

A. Parallel

B. Intersecting

C. Skew

Working. The directions are not multiples of each other; solving rows 1 and 2 gives values that also satisfy row 3 — so the lines intersect.

Answer. **B** — Intersecting

(b) Hence the point of intersection

The work is already done: the λ that classified the lines is the λ that locates the point. Substitute it into L_1. Substituting μ into L_2 must give the same point — a free check.

Examiner Insight

State the verification explicitly. It carries a mark, and it is the most commonly skipped one in this topic.

Worked — part (b)

Find the point of intersection.

Working. $\begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix} + 0 \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix}$; $\mu = 2$ in L_2 gives the same point ✓

Answer. $\begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix}$

(c) And the angle between them

The angle between two lines is the angle between their **directions** — where they sit is irrelevant:

$$\cos\theta = \frac{|\mathbf{d}_1 \cdot \mathbf{d}_2|}{|\mathbf{d}_1||\mathbf{d}_2|}.$$

The modulus gives the acute angle: a line has no direction of travel, so $\mathbf{d}$ and $-\mathbf{d}$ describe the same line.

Examiner Insight

The root-level angle formula, plus modulus signs. That is the only difference.

Worked — part (c)

Find the acute angle between L_1 and L_2, in degrees to 1 d.p.

Working. $\theta = 42.392°$.

Answer. 42.4

7 Session 7 — Distance problems with lines

AHL 3.14 • 8 exam-question parts

7.1 The closest point on a line

A point, a line, and the perpendicular between them. One idea — set a dot product to zero — carries every part, reflection included.

The shape of the question

The point $P(11,\ 8,\ -1)$ and the line $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ are given. N is the point on the line closest to P.

(a) Find the parameter

The closest point is where $\overrightarrow{PN}$ is **perpendicular to the direction**:

$$\overrightarrow{PN} \cdot \mathbf{d} = 0.$$

Write N as $\mathbf{a} + \lambda\mathbf{d}$, form $\overrightarrow{PN} = N - \mathbf{p}$, and the condition becomes one equation in one unknown.

Examiner Insight

Every 'shortest distance from a point to a line' question is this, whatever the wording around it.

Worked — part (a)

Find the value of λ at N.

Working. Solving $\overrightarrow{PN} \cdot \mathbf{d} = 0$ gives $\lambda = 3$.

Answer. 3

(b) Hence the foot of the perpendicular

Substitute that λ back into the line. This is the three-beat that runs through the whole strand: **find the parameter, substitute back, measure.**

Examiner Insight

Check it: $\overrightarrow{PN}$ dotted with the direction should come out exactly zero.

Worked — part (b)

Find the coordinates of N.

Working. $N = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \\ -5 \end{pmatrix}$, and $\overrightarrow{PN} \cdot \mathbf{d} = 0$ ✓

Answer. $\begin{pmatrix} 8 \\ 9 \\ -5 \end{pmatrix}$

(c) And the shortest distance

The shortest distance is just $|\overrightarrow{PN}|$ — the magnitude root skill, applied to the displacement you now have. The perpendicular really is shortest: every other point on the line is further away.

Examiner Insight

If the question asks for the minimum distance *from the origin*, P is simply $\mathbf{0}$ and nothing else changes.

Worked — part (c)

Find the shortest distance from P to the line, to 3 s.f.

Working. $\overrightarrow{PN} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, so the distance is $\sqrt{26} = 5.099$.

Answer. 5.10

(d) And the reflection of P in the line

N is the **midpoint** of P and its mirror image P' — that is what reflecting in the line means. Rearranging $N = \frac{1}{2}(P + P')$ gives

$$P' = 2N - \mathbf{p}.$$

One line of work, once you have the foot.

Examiner Insight

Check it in one step: the midpoint of P and P' must come back to N. Reflecting in a **plane** is the same sentence with the foot of the perpendicular to the plane.

Worked — part (d)

Find the reflection of P in the line.

Working. $P' = 2N - \mathbf{p} = 2\begin{pmatrix}8\\9\\-5\end{pmatrix} - \begin{pmatrix}11\\8\\-1\end{pmatrix} = \begin{pmatrix}5\\10\\-9\end{pmatrix}$. Midpoint check: $\frac{1}{2}(P+P') = \begin{pmatrix}8\\9\\-5\end{pmatrix}$ ✓

Answer. $\begin{pmatrix}5\\10\\-9\end{pmatrix}$

Common Pitfall

That is N, the foot of the perpendicular — only **half way** to the image. Go the same distance again: $P' = 2N - \mathbf{p}$.

Common Pitfall

The 2 multiplies N, not P. N is the midpoint, so $P' = 2N - \mathbf{p}$.

7.2 Two skew lines, and how far apart

The Section B finale. Two lines that never meet still have a closest approach, and the vector perpendicular to both is what measures it.

The shape of the question

$L_1 : \ \mathbf{r} = \begin{pmatrix}3\\-3\\-1\end{pmatrix} + \lambda\begin{pmatrix}-2\\1\\-2\end{pmatrix}$ and $L_2 : \ \mathbf{r} = \begin{pmatrix}3\\12\\-1\end{pmatrix} + \mu\begin{pmatrix}-1\\-1\\2\end{pmatrix}$. The two lines are skew.

(a) A direction perpendicular to both

The shortest route between two skew lines is the one that meets **both** at right angles. A vector along that route is therefore perpendicular to both directions — which is exactly

$$\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2.$$

Every later part hangs off this one vector.

Examiner Insight

Cancel any common factor now. A smaller $\mathbf{n}$ makes the arithmetic in the next part much lighter, and the direction is all that matters.

Worked — part (a)

Find a vector perpendicular to both L_1 and L_2.

Working. $\mathbf{d}_1 \times \mathbf{d}_2 = \begin{pmatrix}-2\\1\\-2\end{pmatrix} \times \begin{pmatrix}-1\\-1\\2\end{pmatrix} = \begin{pmatrix}0\\6\\3\end{pmatrix}$, i.e. $\begin{pmatrix}0\\2\\1\end{pmatrix}$.

Answer. $\begin{pmatrix}0\\2\\1\end{pmatrix}$

Common Pitfall

The sum lies in the plane of the two directions, not perpendicular to it. Perpendicular to **both** is the cross product.

(b) Where the common perpendicular meets L_1

Now locate the two ends of that shortest route. Write the two points as $\mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\mathbf{a}_2 + \mu\mathbf{d}_2$; the vector between them must be **parallel to n**, which gives two equations in λ and μ. Solve, and substitute back.

Examiner Insight

Equivalently, that joining vector is perpendicular to *both* directions — two dot products set to zero. Same pair of equations, whichever way you set it up.

Worked — part (b)

Find the point on L_1 closest to L_2.

Working. $\lambda = 1$, so the point is $\begin{pmatrix}3\\-3\\-1\end{pmatrix} + \begin{pmatrix}-2\\1\\-2\end{pmatrix} = \begin{pmatrix}1\\-2\\-3\end{pmatrix}$.

Answer. $\begin{pmatrix}1\\-2\\-3\end{pmatrix}$

(c) And where it meets L_2

The other end, the same way. The two feet are the **only** pair of points, one on each line, whose joining vector is perpendicular to both — and the distance between them is the answer to part (b), which is a free check on all of it.

Examiner Insight

Check it: subtract the two points and confirm the result is a multiple of **n**, and that its length matches the distance you already found.

Worked — part (c)

Find the point on L_2 closest to L_1.

Working. $\mu = 2$ gives $\begin{pmatrix}3\\12\\-1\end{pmatrix} + 2\begin{pmatrix}-1\\-1\\2\end{pmatrix} = \begin{pmatrix}1\\10\\3\end{pmatrix}$. Subtracting the two feet gives $\begin{pmatrix}0\\12\\6\end{pmatrix}$ — a multiple of **n**, of length 13.416 ✓

Answer. $\begin{pmatrix} 1 \\ 10 \\ 3 \end{pmatrix}$

(d) The shortest distance between them

Nothing new to set up. You located both ends of the shortest route by making the joining vector perpendicular to each direction, so the distance is just the length of the vector between them — the magnitude skill, one more time.

Examiner Insight

A worthwhile check before you take the magnitude: $\overrightarrow{F_1F_2}$ must be a multiple of $\mathbf{n}$. If it is not, the two parameters are wrong, not the arithmetic here.

Worked — part (d)

Hence find the shortest distance between L_1 and L_2, to 3 s.f.

Working. $\overrightarrow{F_1F_2} = \begin{pmatrix} 0 \\ 12 \\ 6 \end{pmatrix}$, which is a multiple of $\mathbf{n} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ ✓Distance $= |\overrightarrow{F_1F_2}| = 13.416$.

Answer. 13.4

Common Pitfall

That is the distance between the two given points A_1 and A_2, not between the lines. Those points are almost never the closest pair.

8 Session 8 — The vector (cross) product, and planes

AHL 3.16 • 5 basics • 7 exam-question parts

8.1 The vector product

Unlike the dot product this returns a **vector**, perpendicular to both inputs:

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}.$$

The middle row is the reversed one — that is the minus sign in the determinant, and where nearly every sign error happens.

Examiner Insight

Check it by dotting the answer with both inputs. Both should give exactly zero.

Worked Example

Find $\begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$.

Working. $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 0 \\ -8 \\ 0 \end{pmatrix}$. Dotting it with either input gives 0 ✓

Worked Example

Find $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$.

Working. $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Dotting it with either input gives 0 ✓

8.2 What the cross product means

Two facts, and every cross-product question uses one of them. **Direction**: $\mathbf{u} \times \mathbf{v}$ is perpendicular to *both* — which is where every plane's normal comes from. **Size**: $|\mathbf{u} \times \mathbf{v}|$ is the **area of the parallelogram** they span, so half of it is the area of the triangle.

Examiner Insight

$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$ — the sine partner to the dot product's cosine. Parallel vectors give zero area, which is another way of testing parallel.

Worked Example

Find the area of the parallelogram with $\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$ as two of its sides, to 3 s.f.

Working. $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} -5 \\ -1 \\ -12 \end{pmatrix}$, so the area is $\left| \begin{pmatrix} -5 \\ -1 \\ -12 \end{pmatrix} \right| = 13.038$.

Worked Example

Find the area of the parallelogram with $\begin{pmatrix} -3 \\ -2 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$ as two of its sides, to 3 s.f.

Working. $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} -12 \\ 14 \\ 2 \end{pmatrix}$, so the area is $\left| \begin{pmatrix} -12 \\ 14 \\ 2 \end{pmatrix} \right| = 18.547$.

8.3 The right-hand rule

The cross product is perpendicular to both — but *which* of the two opposite directions? Point your right hand along $\mathbf{u}$, curl your fingers toward $\mathbf{v}$, and your **thumb** gives $\mathbf{u} \times \mathbf{v}$. On the axes this cycles:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Examiner Insight

Order matters: $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$. Swapping the two vectors flips the normal, which flips the sign of a plane's equation — the same plane, written the other way round.

Worked Example

Using the right-hand rule, find $\mathbf{j} \times \mathbf{i}$.

Working. That is backwards round the cycle, so the answer is negative: $\mathbf{j} \times \mathbf{i} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$.

Worked Example

Using the right-hand rule, find $\mathbf{i} \times \mathbf{k}$.

Working. That is backwards round the cycle, so the answer is negative: $\mathbf{i} \times \mathbf{k} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$.

8.4 A plane from two directions

A line needs one direction; a plane needs **two**. Starting from a point $\mathbf{a}$ and two non-parallel directions $\mathbf{b}$ and $\mathbf{c}$ lying in it,

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$$

sweeps out the whole plane — two dials instead of one. Cross those two directions and you get the normal, which is the bridge to the other two forms.

Examiner Insight

The two directions must not be parallel, or they sweep out a line and not a plane. Any two independent directions in the plane will do — three points A, B, C give you $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Worked Example

A plane is $\mathbf{r} = \begin{pmatrix} -2 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix}$. Find a normal vector to it.

Working. $\mathbf{b} \times \mathbf{c} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$, i.e. $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$.

Worked Example

A plane is $\mathbf{r} = \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -2 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$. Find a normal vector to it.

Working. $\mathbf{b} \times \mathbf{c} = \begin{pmatrix} -3 \\ -2 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ -7 \end{pmatrix}$, i.e. $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

8.5 The forms of a plane

A plane is pinned down by **one normal and one point on it**. Scalar-product form says every point of the plane has the same projection onto the normal:

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}.$$

Writing $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and multiplying out turns that straight into **Cartesian** form, $ax + by + cz = d$ — the components of $\mathbf{n}$ *are* the coefficients.

Examiner Insight

So you can read a normal straight off any Cartesian equation, and that is usually the first thing a plane question needs.

Worked Example

A plane is $4x + 2y = 8$. Write down a normal vector to it.

Working. The coefficients give $\mathbf{n} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$ — or any scalar multiple of it.

Worked Example

A plane is $-4x - 2y + 3z = -13$. Write down a normal vector to it.

Working. The coefficients give $\mathbf{n} = \begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$ — or any scalar multiple of it.

8.6 A triangle becomes a plane

Three points define a triangle — and the same three points define a plane. One cross product does both jobs.

The shape of the question

The points $A(-3, -2, 1)$, $B(-6, 1, -1)$ and $C(-6, 1, 3)$ are the vertices of a triangle.

(a) The cross product

Both displacements must start from the **same vertex**. Then $\overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to both — and so to the whole plane of the triangle.

Examiner Insight

Check by dotting with both inputs. Both should give zero.

Worked — part (a)

Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

Working. $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 12 \\ 12 \\ 0 \end{pmatrix}$.

Answer. $\begin{pmatrix} 12 \\ 12 \\ 0 \end{pmatrix}$

(b) Hence the area

$|\mathbf{u} \times \mathbf{v}|$ is the area of the **parallelogram** spanned by the two vectors, so a triangle is half of it:

$$\text{area} = \tfrac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|.$$

Examiner Insight

Forgetting the $\frac{1}{2}$ is the single most common lost mark in this type.

Worked — part (b)

Hence find the area of triangle ABC, to 3 s.f.
Working. $\frac{1}{2}\sqrt{288} = 8.485$.
Answer. 8.48

Common Pitfall

That is the *parallelogram*. A triangle is half of it.

(c) And the plane through them

The same cross product is the plane's **normal**. Then

$$ax + by + cz = d, \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{n}, \quad d = \mathbf{n} \cdot \mathbf{a}.$$

This is the cross product's principal job in the course.

Examiner Insight

Fix d with any one of the three points — all three give the same value, which is a free check.

Worked — part (c)

Find the Cartesian equation of the plane containing A, B and C.
Working. $d = -60$, so $12x + 12y = -60$. Any scalar multiple is the same plane.
Answer. $12x + 12y = -60$

8.7 The parallelogram

Three vertices given, the fourth to find, then the diagonals, the area and the plane it sits in. The standard Section B opener, taken from three points to a plane in four steps.

The shape of the question

$ABCD$ is a parallelogram with $A(4,\ -5,\ -3)$, $B(5,\ -1,\ -4)$ and $C(-2,\ -5,\ 2)$, where D is diagonally opposite B.

(a) The fourth vertex

In a parallelogram opposite sides are equal **as vectors**, so $\overrightarrow{AD} = \overrightarrow{BC}$. That rearranges to

$$\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}.$$

No diagram needed — the vector equation says which vertex is which.

Examiner Insight

Watch the lettering. $ABCD$ goes round the shape, so D is opposite B — pairing the wrong vertices is the usual lost mark here.

Worked — part (a)

Find the coordinates of D.

Working. $\mathbf{d} = \begin{pmatrix} 4 \\ -5 \\ -3 \end{pmatrix} + \begin{pmatrix} -2 \\ -5 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} -3 \\ -9 \\ 3 \end{pmatrix}$.

Answer. $\begin{pmatrix} -3 \\ -9 \\ 3 \end{pmatrix}$

Common Pitfall

That places the fourth vertex opposite the wrong one. In $ABCD$ the order goes round the shape, so D is opposite B: $\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}$.

(b) Where the diagonals cross

The diagonals of a parallelogram **bisect each other**, so they cross at the midpoint of either one. AC is the easier: average $\mathbf{a}$ and $\mathbf{c}$ and you are done.

Examiner Insight

You could average $\mathbf{b}$ and $\mathbf{d}$ instead and must get the same point — a free check that part (a) was right.

Worked — part (b)

The diagonals $[AC]$ and $[BD]$ meet at E. Find the coordinates of E.

Working. $\frac{1}{2}\left(\begin{pmatrix} 4 \\ -5 \\ -3 \end{pmatrix} + \begin{pmatrix} -2 \\ -5 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ -5 \\ -0.5 \end{pmatrix}$ — and $\frac{1}{2}(\mathbf{b} + \mathbf{d})$ gives the same point ✓

Answer. $\begin{pmatrix} 1 \\ -5 \\ -0.500 \end{pmatrix}$

(c) Its area

Two sides from the same vertex span the shape, so its area is $\left|\overrightarrow{AB} \times \overrightarrow{AD}\right|$ — or equivalently $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$ using the diagonal, since the diagonal and the far side differ by a vector already counted.

Examiner Insight

No halving here. Halve only for a triangle; this is the whole parallelogram.

Worked — part (c)

Find the area of parallelogram $ABCD$, to 3 s.f.

Working. $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 20 \\ 1 \\ 24 \end{pmatrix}$, so the area is 31.257.

Answer. 31.3

Common Pitfall

That is the triangle ABC. The parallelogram is twice it — no $\frac{1}{2}$ this time.

(d) And the plane it lies in

All four vertices lie in one plane, and the cross product from the last part is already its **normal**. Use any vertex for the constant and the Cartesian equation follows in a line.

Examiner Insight

You have done the hard part already. Recognising that the area computation hands you the normal for free is what makes these questions quick.

Worked — part (d)

Find the Cartesian equation of the plane containing $ABCD$.

Working. $d = \mathbf{n} \cdot \mathbf{a} = 3$, so the plane is $20x + y + 24z = 3$.

Answer. $20x + y + 24z = 3$

9 Session 9 — Intersections and angles with planes

AHL 3.16 • 14 exam-question parts

9.1 A line meets a plane

Where a line cuts a plane, at what angle, and how far off it a point sits. Contains the sine-versus-cosine distinction the IB tests most years.

The shape of the question

The plane $\Pi : -6x + 6y + 7z = -26$, the line $\mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ and the point $Q(8, -18, -16)$ are given.

(a) Where the line crosses

Substitute the line's parametric coordinates into the plane equation. It collapses to **one equation in** λ; solve, then put λ back. If the λ terms cancel entirely, the line is parallel to the plane.

Examiner Insight

The same three-beat again: reduce to one unknown, solve, substitute back.

Worked — part (a)

Find the point where the line meets Π.

Working. $\lambda = 0$, so the point is $\begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}$.

Answer. $\begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}$

(b) At what angle — and why sine

This one uses **sine**:

$$\sin\theta = \frac{|\mathbf{d}\cdot\mathbf{n}|}{|\mathbf{d}||\mathbf{n}|}.$$

The formula naturally finds the angle to the **normal**; the angle to the *plane* is its complement, and $\sin^{-1}$ makes that switch for you.

Examiner Insight

Line-to-plane: sine. Plane-to-plane: cosine. Say it back before you press the button.

Worked — part (b)

Find the angle between the line and Π, in degrees to 1 d.p.

Working. $\theta = 21.556°$.

Answer. 21.6

Common Pitfall

You used $\cos^{-1}$, which gives the angle to the **normal**. The angle to the plane is its complement — use $\sin^{-1}$.

(c) How far Q is from the plane

No formula. Walk from Q along the normal until you land on the plane: the point $\mathbf{q} + t\mathbf{n}$ lies on Π exactly when

$$\mathbf{n} \cdot (\mathbf{q} + t\mathbf{n}) = d,$$

which is one linear equation in t. Solve it, and $t\mathbf{n}$ is the step that takes you there — so the distance is $|t|\,|\mathbf{n}|$.

Examiner Insight

Solve for t and keep it. The same t gives you the foot of the perpendicular and the reflection, so it is worth writing down rather than folding into a distance.

Worked — part (c)

Find the distance from Q to Π, to 3 s.f.

Working. $\mathbf{n} \cdot \mathbf{q} = -268$ and $|\mathbf{n}|^2 = 121$, so $-268 + 121t = -26$ gives $t = 2$. Distance $= |t|\,|\mathbf{n}| = 2 \times 11 = 22$.

Answer. 22

(d) The reflection of Q in the plane

The same step, taken twice. With the t you just solved, the foot is $F = \mathbf{q} + t\mathbf{n}$; carry on the same distance again and you arrive at the image,

$$Q' = \mathbf{q} + 2t\mathbf{n}.$$

Equivalently F is the midpoint of Q and Q' — which is how you check it.

Examiner Insight

Keep the sign of t. A modulus here would send the image to the wrong side of the plane.

Worked — part (d)

Find the reflection of $Q(8, -18, -16)$ in the plane Π.

Working. $F = \mathbf{q} + t\mathbf{n} = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix}$, so $Q' = \mathbf{q} + 2t\mathbf{n} = \begin{pmatrix} -16 \\ 6 \\ 12 \end{pmatrix}$. The midpoint of Q and Q' is $\begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix}$, the foot again ✓

Answer. $\begin{pmatrix} -16 \\ 6 \\ 12 \end{pmatrix}$

Common Pitfall

That is the foot of the perpendicular — you stopped at the plane. Take the same step again to reach the image.

(e) And two planes together

The angle between two planes is the angle between their **normals** — and here it *is* cosine:

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}.$$

Both objects are already represented by perpendicular vectors, so no complement is needed.

Examiner Insight

That is the whole rule: two normals $\Rightarrow$ cos; one direction and one normal $\Rightarrow$ sin.

Worked — part (e)

A second plane has normal $\begin{pmatrix} -2 \\ 6 \\ 9 \end{pmatrix}$. Find the acute angle between the two planes, in degrees to 1 d.p.

Working. $\theta = 23.458°$.

Answer. 23.5

9.2 A line that never meets a plane

Substituting a line into a plane usually gives one value of λ. When the λ terms cancel instead, something has gone right, not wrong — and which of two things it is depends on the constant.

The shape of the question

The plane Π has equation $-4x - 2y + 2z = 0$, and the line L is $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix}$.

(a) Why lambda disappears

Substitute the line into the plane and the coefficient of λ is $\mathbf{d} \cdot \mathbf{n}$. If that is **zero**, λ vanishes from the equation entirely and there is nothing left to solve — the line runs along the plane rather than through it.

Examiner Insight

So the very first thing to compute is $\mathbf{d} \cdot \mathbf{n}$. Non-zero means one crossing point; zero means parallel, and only then do you ask whether it lies *in* the plane.

Worked — part (a)

Find $\mathbf{d} \cdot \mathbf{n}$ for this line and plane.

Working. $\begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix} = 0$ — the direction is perpendicular to the normal, so the line is parallel to the plane.

Answer. 0

(b) Parallel, or lying inside it?

Perpendicular to the normal only says the line is **parallel** to the plane. Two things fit that: the line floats above it, or it lies inside it. Test a single point of the line in the plane's equation — one substitution settles it.

Examiner Insight

Examiners give the mark for the test, not the conclusion. Show the substitution and the mismatch.

Worked — part (b)

Substitute the point $(2, 3, 2)$ into $-4x - 2y + 2z = d$ — what value of d does it give?

Working. $\mathbf{n} \cdot \mathbf{a} = -10$, but the plane needs 0. They differ, so the point is not on the plane and the line is parallel to it, not inside it.

Answer. -10

Common Pitfall

That is the plane's own constant. You are computing what the **line's** point gives — if they matched, the line would lie in the plane.

(c) How far off it sits

Because the line is parallel, every one of its points is the same distance from the plane — so pick the point you were handed and do it the long way. Step from **a** along the normal and ask when you land on the plane:

$$\mathbf{n} \cdot (\mathbf{a} + t\mathbf{n}) = d.$$

One equation, one unknown. The distance is then $|t|\,|\mathbf{n}|$.

Examiner Insight

This only works because the line is parallel. For a line that crosses the plane the distance is zero, and asking for it makes no sense.

Worked — part (c)

Find the distance between L and Π, to 3 s.f.

Working. $-10 + 24t = 0$ gives $t = 0.417$. Distance $= |t|\,|\mathbf{n}| = 0.417 \times 4.899 = 2.041$.

Answer. 2.04

(d) And a line that does lie in it

Change only the point. Same direction, so still parallel — but now the point satisfies the plane's equation, and the line is **contained** in the plane. Every one of its points is a solution, so the line and plane meet in infinitely many places rather than none.

Examiner Insight

Two conditions, both needed: $\mathbf{d} \cdot \mathbf{n} = 0$ **and** the point on the plane. Showing only one of them is a half-answer.

Worked — part (d)

A second line M has the same direction but passes through $(-2,\ 1,\ -3)$. Show M lies in Π by finding $\mathbf{n}\cdot$ that point.

Working. $\mathbf{n} \cdot \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} = 0$, matching the plane exactly. With the direction already perpendicular to $\mathbf{n}$, every point of M satisfies the equation.

Answer. 0

9.3 Three planes together

Two planes meet in a line; a third either pins that line to a point, contains it, or misses it. Reading which of those has happened is the whole skill.

The shape of the question

Three planes are given: $\Pi_1 : 3y = 6$, $\Pi_2 : -2x - 3y + 2z = 4$ and $\Pi_3 : -3x + 3y - 3z = 3$.

(a) Where two planes meet

Two non-parallel planes always meet in a **line**. That line lies in both planes, so it is perpendicular to **both** normals — which is exactly what the cross product delivers:

$$\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2.$$

Examiner Insight

You never need to solve anything to get the *direction*. Solving only comes in when you want a point on the line.

Worked — part (a)

Find a direction vector for the line where Π_1 and Π_2 meet.

Working. $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \times \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix}$, i.e. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ after cancelling.

Answer. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Common Pitfall

Adding the normals gives a vector in the plane *of the normals*, not along the line. The line must be perpendicular to both normals, so it is the **cross** product.

(b) Where all three meet

Three planes with independent normals meet at exactly **one point**. Solve the three equations together — eliminate one variable between two pairs, then back-substitute. A GDC will do it, but the elimination is what the method marks are for.

Examiner Insight

Always substitute your point back into all three equations. It costs seconds and catches every arithmetic slip.

Worked — part (b)

Find the point where all three planes meet.

Working. Solving the system gives $\begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$, and it satisfies all three equations ✓

Answer. $\begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$

(c) Hence the line where two planes meet

Part (a) gave the **direction** of that line and part (b) gave a **point** that lies on every one of the planes — so it certainly lies on the first two. Direction plus point is a line, and no further solving is needed.

Examiner Insight

Any point of the line is acceptable, and any multiple of the direction. If you had not already found P, you would set one variable free — put $z = 0$, say — and solve the two plane equations for the other two.

Worked — part (c)

Write down a vector equation for the line where Π_1 and Π_2 meet.

Working. $\mathbf{r} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ — and substituting it into either plane's equation satisfies it for every λ ✓

Answer. $\mathbf{r} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(d) When the third plane contains the line

Now replace Π_3. If the new normal is a **combination of the first two**, the third plane can no longer pin the line down — it either contains that line or misses it entirely. Which one depends on the constant.

Examiner Insight

This is what an exam means by 'for what value of k does the system have infinitely many solutions'. The normals decide *whether* it can happen; the constant decides *if it does*.

Worked — part (d)

Π_3 is now replaced by $-2x + 2z = 10$, which is $\Pi_1 + \Pi_2$ exactly. What is the solution set?

A. A single point

B. A line

C. No solution

Working. The third equation is the sum of the first two, so elimination turns it into $0 = 0$. Two independent equations in three unknowns leave one free parameter — a **line**.

Answer. **B** — A line

(e) When it misses

Same normals, different constant. The three planes are still pairwise non-parallel, so each pair still meets in a line — but those three lines are now parallel and distinct. The planes bound a **triangular prism** and nothing lies on all three.

Examiner Insight

No two planes being parallel does **not** mean there is a solution. The prism is the case students forget, and it is the one examiners set.

Worked — part (e)

Instead Π_3 is $-2x + 2z = 13$ — the same normal, constant changed by 3. What is the solution set now?

A. A single point

B. A line

C. No solution

Working. Adding the first two gives $-2x + 2z = 10$, but Π_3 says the same left-hand side equals 13. Elimination reaches $0 = 3$, which is impossible — **no solution**, a triangular prism.

Answer. **C** — No solution

10 Session 10 — Applications and synthesis

AHL 3.12–3.16 • 9 exam-question parts

10.1 Two ships

A line equation with time as the parameter. The shape the IB likes to end a Section B question with.

The shape of the question

Ship A starts at $\begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$ with constant velocity $\begin{pmatrix} -9 \\ -12 \\ 0 \end{pmatrix}$ km h^{-1}. Ship B starts at $\begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}$ with constant velocity $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ km h^{-1}.

(a) Position at a later time

Constant velocity is a line with time as the parameter:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

Everything you know about lines transfers — the 'direction vector' is now the velocity.

Examiner Insight

Watch units. Velocity in km h^{-1} pairs with t in hours; a time in minutes must be converted first.

Worked — part (a)

Find the position vector of ship A at $t = 5$ hours.

Working. $\begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} -9 \\ -12 \\ 0 \end{pmatrix} = \begin{pmatrix} -42 \\ -54 \\ 0 \end{pmatrix}$.

Answer. $\begin{pmatrix} -42 \\ -54 \\ 0 \end{pmatrix}$

(b) Its speed

Velocity is a vector; **speed** is its magnitude — a single positive number. The magnitude root skill, one last time.

Examiner Insight

If a question says 'find the speed' and your answer has three rows, re-read it.

Worked — part (b)

Find the speed of ship A.

Working. $|\mathbf{v}| = \sqrt{225} = 15$ km h^{-1}.

Answer. 15

(c) When they are closest

Form the **separation** $\mathbf{s}(t) = \mathbf{r}_A - \mathbf{r}_B$ and minimise its magnitude. Minimising $|\mathbf{s}|^2$ avoids the square root and has its minimum at the same t:

$$t = -\frac{\mathbf{s}_0 \cdot \Delta\mathbf{v}}{|\Delta\mathbf{v}|^2}.$$

Underneath, this is the perpendicularity condition again.

Examiner Insight

A collision is the special case where the separation reaches exactly **0**. Closest approach is the commoner question.

Worked — part (c)

Find the time at which the two ships are closest, to 3 s.f.

Working. $t = -\dfrac{-74}{218} = 0.339$ hours.

Answer. 0.339

(d) And how close they get

Put that t back into the separation and take its magnitude. **Find the parameter, substitute back, measure** — the same three-beat as the closest point on a line, and as the line meeting a plane.

Examiner Insight

If a question in this strand feels new, ask which parameter you are solving for. It is usually one of these three.

Worked — part (d)

Find the distance between the ships at that moment, to 3 s.f.

Working. The separation then has magnitude 2.98 km.

Answer. 2.98

10.2 Do they collide?

Two aircraft whose paths cross. Crossing is not colliding — the question is whether they are ever at the crossing point at the same moment, and that is the distinction examiners are testing.

The shape of the question

Two aircraft A and B have position vectors, relative to an origin O on the ground, of

$$\mathbf{r}_A = \begin{pmatrix} -24 \\ -8 \\ -8 \end{pmatrix} + t \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}, \qquad \mathbf{r}_B = \begin{pmatrix} -12 \\ -4 \\ 6 \end{pmatrix} + t \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix}$$

where t is the time in minutes, distances are in km, and x, y, z point east, north and vertically up.

(a) The bearing A is flying on

A bearing is measured **clockwise from north**, so with x east and y north it is the angle of $\begin{pmatrix} x \\ y \end{pmatrix}$ from the y-axis, not the x-axis:

$$\theta = \arctan\left(\frac{x}{y}\right),$$

adjusted into 0°–360°. The vertical component plays no part — a bearing is a compass direction on the ground.

Examiner Insight

Give three figures: 070°, not 70°. And check the quadrant — a calculator's arctan will hand you an angle in the wrong half of the compass if $y < 0$.

Worked — part (a)

Find the three-figure bearing on which aircraft A is flying, to 3 s.f.

Working. $\arctan\left(\frac{4}{1}\right)$, placed in the correct quadrant, gives $75.964°$.

Answer. 76.0

Common Pitfall

That is measured anticlockwise from **east** — the standard maths convention. A bearing is clockwise from **north**, so the roles of x and y swap.

(b) Which one is faster

The vector multiplying t is the **velocity**; its magnitude is the **speed**. Direction lives in the vector, size lives in its magnitude — so comparing two aircraft is just comparing two magnitudes.

Examiner Insight

Speed is a length, so it is never negative and never a vector. If a question says 'show A is faster', computing both magnitudes and comparing them *is* the whole answer.

Worked — part (b)

Find the speed of aircraft B, in km min^{-1}, to 3 s.f.

Working. $\left|\begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix}\right| = 6.164$ km min^{-1}, against A's 4.583.

Answer. 6.16

Common Pitfall

That is A's speed. Check which velocity vector belongs to which aircraft.

(c) The angle between their paths

The angle between two flight paths is the angle between their **velocity** vectors — where the aircraft happen to be does not matter. Taking the modulus of the dot product gives the acute angle, which is what is asked for.

Examiner Insight

This is the same formula as any 'angle between two lines'. Velocity is simply the direction vector wearing a different name.

Worked — part (c)

Find the acute angle between the two flight paths, in degrees to 1 d.p.

Working. $\cos\theta = \dfrac{|23|}{4.583 \times 6.164}$, giving $\theta = 35.493°$.

Answer. 35.5

(d) Where the paths cross

Here is the trap. To find where the **paths** cross you must let each aircraft have its **own** parameter — $\mathbf{r}_A$ at time s and $\mathbf{r}_B$ at time t — because they need not be there together. Using one t for both asks a different question entirely.

Examiner Insight

Solve two of the three rows for s and t, then check the third. If it fails, the paths do not cross at all and the aircraft are on skew courses.

Worked — part (d)

The two flight paths cross at a point P. Find the coordinates of P.

Working. $s = 6$ and $t = 2$ both give $\begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}$, so the paths cross there.

Answer. $\begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}$

(e) So do they collide?

Two paths crossing says nothing about a collision. A collision needs **the same point at the same time** — one value of t satisfying all three rows simultaneously. Here the two aircraft reach P at different times, so they miss each other.

Examiner Insight

This is the single most examined idea in vector kinematics. 'Show that the aircraft do not collide' is answered by finding the two times and stating they differ — not by finding the distance.

Worked — part (e)

Find how long after A passes through P that B passes through it, in minutes.

Working. A reaches P at $t = 6$ and B at $t = 2$, a gap of 4 minutes. The paths cross, the aircraft do not.

Answer. 4

Common Pitfall

Zero would mean they arrive together — a collision. Solve for each aircraft's own time at P and compare: they are different, which is exactly why there is no collision.

Where each piece lives

A one-page map of the strand, so you can find anything quickly.

Session	Basics	Exam-question types
1	Notation, Adding and scaling, AB = b − a, The midpoint	Geometry without coordinates
2	Magnitude, Unit vector, Parallel vectors, Parallel and anti-parallel	Three points
3	The scalar product, What the sign of the dot product tells you, Angle between vectors, Perpendicular	—
4	—	Find the value of k, Show that it is always true
5	The forms of a line, What the parameter does	The equation of a line
6	How two lines can sit	Two lines in space
7	—	The closest point on a line, Two skew lines, and how far apart
8	The vector product, What the cross product means, The right-hand rule, A plane from two directions, The forms of a plane	A triangle becomes a plane, The parallelogram
9	—	A line meets a plane, A line that never meets a plane, Three planes together
10	—	Two ships, Do they collide?