PHOTON ACADEMY · IB MATH AI HL CHEATSHEET

Eigenvalues, Eigenvectors & Markov Chains

AHL 1.15

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FORMULAS YOU MUST KNOW

$A\mathbf{v} = \lambda\mathbf{v}$ with $\mathbf{v} \neq \mathbf{0}$ — applying A just scales $\mathbf{v}$ by λ

Definition:

$\det(A - \lambda I) = 0$ — the equation whose roots are the eigenvalues

Characteristic eq.:

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$

2×2 form:

$\lambda^2 - (\mathrm{tr}\,A)\lambda + \det A = 0$ — sum of roots $=$ $\mathrm{tr}\,A$, product $= \det A$

In words:

$\lambda = \dfrac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$

Quadratic formula:

for $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ the equation $(a-\lambda)^2 - b^2 = 0$ is a difference of squares $\Rightarrow \lambda = a \pm b$

Symmetric shortcut:

$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \Rightarrow \lambda = 2 \pm 1$, so $\lambda = 3$ or $\lambda = 1$

Example:

write $\mathbf{v} = \begin{pmatrix} 1 \\ k \end{pmatrix}$; the top row of $(A - \lambda I)\mathbf{v} = \mathbf{0}$ gives $(a - \lambda) + bk = 0$

Set-up:

$k = \dfrac{\lambda - a}{b}$, so $\mathbf{v} = \begin{pmatrix} 1 \\ \frac{\lambda - a}{b} \end{pmatrix}$ (or any scalar multiple)

Component:

for $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$: $\lambda = 3 \Rightarrow k = 1 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda = 1 \Rightarrow k = -1 \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Example:

$A = PDP^{-1}$, where $P = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

Diagonalise:

$A^n = PD^nP^{-1}$ with $D^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$ — just raise the eigenvalues

n-th power:

KEY METHODS

- The trace $a + d$ is the sum of the eigenvalues and $\det A = ad - bc$ is their product. Use these as an instant check once you have your two values — but neither number is itself an eigenvalue.
- A symmetric matrix ($b = c$) always has two real eigenvalues, because $(a-d)^2 + 4b^2 \geq 0$. When the diagonal entries are equal, factor by difference of squares and skip the formula entirely — the larger value $a + b$ is the one that controls long-run growth.

EXAM TRAPS TO AVOID

- You subtract λ only on the leading diagonal: $A - \lambda I = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$. Subtracting it from every entry is the classic set-up error.
- The two rows of $(A - \lambda I)$ are dependent (that is exactly why $\det = 0$). Use one row only — trying to solve both as independent equations forces $\mathbf{v} = \mathbf{0}$ and loses the eigenvector.
- In the triangular formula the off-diagonal entry subtracts, $q^n - p^n$ — a very common slip is to add.

- In AI you may simply enter the matrix and let the GDC return the eigenvalues. Reach for the algebra when a question says "show that" or gives the matrix in terms of a parameter.
- An eigenvector defines a direction, not a fixed length — any non-zero multiple $\begin{pmatrix}2\\2\end{pmatrix}$, $\begin{pmatrix}-1\\-1\end{pmatrix}$ is equally valid. Examiners accept any correct multiple unless a normalisation is demanded.
- For a plain numerical power the GDC gives A^n directly — no diagonalisation needed. Save $A = PDP^{-1}$ for a "show that $A^n = \ldots$" or a "behaviour as $n \to \infty$" question, where the eigenvalues carry the meaning: $|\lambda| < 1$ decays to 0, $\lambda = 1$ holds steady, $|\lambda| > 1$ grows.
- Every entry of T is a probability, so $0 \leq T_{ij} \leq 1$. If a column of your matrix does not add to 1, you have mislabelled a transition — fix it before stepping forward.
- Store T and $\mathbf{s}_0$ on the GDC and compute $T^n\mathbf{s}_0$ in one keystroke — far quicker and safer than multiplying n times by hand. If a state vector holds counts (people, animals), keep the total constant at every step as a self-check.

The bottom-left stays 0, and you raise the eigenvalues to the power n, never the whole matrix entry-by-entry.

- With this convention the columns sum to 1, not the rows, and the update is $\mathbf{s}_{n+1} = T\mathbf{s}_n$ with $\mathbf{s}$ a column vector. Do not transpose T — swapping its rows and columns is the single most common error on the whole topic.
- The order is $T^n\mathbf{s}_0$, with the matrix on the left of the column vector — $\mathbf{s}_0 T^n$ is undefined (or gives nonsense) in this convention.
- The long-run split is almost never $50/50$. It tilts toward whichever state is harder to leave (smaller outflow), so a chain with $a = 0.1$, $b = 0.2$ settles at $\frac{0.2}{0.3} = \frac{2}{3}$ in state 1, not one-half.