Coupled & Second-Order Differential Equations

FORMULAS YOU MUST KNOW

$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy$ — each rate depends on both x and y
Coupled system:

$\dot{\mathbf{X}} = M\mathbf{X}$, with $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
Matrix form:

$\operatorname{tr} M = a + d$ and $\det M = ad - bc$ — the two numbers that decide the behaviour
Trace & determinant:

where $\dot{\mathbf{X}} = \mathbf{0}$; for a linear system this is the origin $(0, 0)$
Equilibrium:

$\det(M - \lambda I) = 0 \;\Rightarrow\; \lambda^2 - (\operatorname{tr} M)\lambda + \det M = 0$
Characteristic eq.:

$\lambda = \dfrac{\operatorname{tr} M \pm \sqrt{(\operatorname{tr} M)^2 - 4\det M}}{2}$ — the larger root takes the $+$ sign
Quadratic formula:

for $M = \begin{pmatrix} m & n \\ n & m \end{pmatrix}$ the roots are simply $\lambda = m \pm n$
Symmetric shortcut:

with real eigenvalues the solution mixes $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$; the largest λ dominates as $t \to \infty$
Long-run:

$\Delta = (\operatorname{tr} M)^2 - 4\det M$
Discriminant:

two real distinct eigenvalues — a node (same sign) or a saddle (opposite signs)
$\Delta > 0$:

a repeated eigenvalue — a degenerate (improper) node
$\Delta = 0$:

a complex pair $\lambda = p \pm qi$ — a spiral ($p \neq 0$) or a centre ($p = 0$)
$\Delta < 0$:

KEY METHODS

- Read the coefficients straight into M row by row: the top row is the $\dot{x}$ equation, the bottom row the $\dot{y}$ equation. Then $\operatorname{tr} M = a + d$ and $\det M = ad - bc$ are all you need to launch the eigenvalue method.
- Everything on this sheet flows from $\operatorname{tr} M$ and $\det M$. They are the sum and product of the eigenvalues, so keep them to hand — they double as an instant check on the two values you find.
- On the GDC you can enter M and read the eigenvalues straight off, or solve $\lambda^2 - (\operatorname{tr} M)\lambda + \det M = 0$ with the polynomial solver. Reach for the

EXAM TRAPS TO AVOID

- You subtract λ on the leading diagonal only: $M - \lambda I = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$. Subtracting it from every entry is the classic set-up error, and it wrecks the characteristic equation.
- A saddle is always unstable, however negative one eigenvalue is — the single positive eigenvalue flings almost every trajectory off to infinity. Only the two special directions along the eigenvectors behave differently.
- Each exponential rides its own eigenvector: $c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$. Pairing λ_1 with $\mathbf{v}_2$ is a fatal

algebra only when a question says "show that" or gives M in terms of a parameter.

- This is just the discriminant $b^2 - 4ac$ you already know, applied to $\lambda^2 - (\operatorname{tr} M)\lambda + \det M = 0$ (so $a = 1$, $b = -\operatorname{tr} M$, $c = \det M$). Its sign alone fixes the type of equilibrium before you compute a single eigenvalue.
- A one-line shortcut: if $\det M < 0$ the eigenvalues are automatically real with opposite signs, so the equilibrium must be a saddle — you need not compute Δ at all.
- One rule covers stability: the equilibrium is stable only if every eigenvalue has a negative real part. For a spiral that means $p = \frac{1}{2} \operatorname{tr} M < 0$; for a node it means both real roots are negative — equivalently $\operatorname{tr} M < 0$ and $\det M > 0$.
- Because every exponential equals 1 at $t = 0$, fitting the initial state is just a linear system $c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = \mathbf{X}(0)$. Solve it on the GDC, or by adding and subtracting when the eigenvectors are the tidy $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

mix-up — label each eigenvalue with its eigenvector the moment you find it.

- The bottom row carries the ODE's coefficients with flipped signs: $\dot{y} = -qx - py$. Forgetting to move $p\dot{x}$ and qx across the equals sign — so dropping those minus signs — is the standard slip when reducing.
- Compute both derivatives from the current point before updating either variable. Feeding a half-updated x_{n+1} into $\dot{y}_n$ is the most common Euler error on a coupled system — it silently corrupts every step that follows.
- $x^* = \dfrac{r}{s}$ comes from the predator equation ($\dot{y} = 0$) and $y^* = \dfrac{p}{q}$ from the prey equation ($\dot{x} = 0$) — each species' level is fixed by the other's parameters. Don't swap them, and always reject the trivial $(0, 0)$.