PHOTON ACADEMY · IB MATH AI HL CHEATSHEET

Affine Transformations

AHL 3.9–3.11

ibtuition.sg · free

FORMULAS YOU MUST KNOW

the new x uses the top row, the new y the bottom row — never entry-by-entry

Row × column:

column 1 is the image of $\begin{pmatrix}1\\0\end{pmatrix}$, column 2 the image of $\begin{pmatrix}0\\1\end{pmatrix}$

Reading M:

centre O, factor k: $\begin{pmatrix}k & 0\\0 & k\end{pmatrix}$ — multiplies both coordinates by k

Enlargement:

factor p: $\begin{pmatrix}p & 0\\0 & 1\end{pmatrix}$ (leaves y unchanged)

Horizontal stretch:

factor q: $\begin{pmatrix}1 & 0\\0 & q\end{pmatrix}$ (leaves x unchanged)

Vertical stretch:

$\begin{pmatrix}p & 0\\0 & q\end{pmatrix}$ — stretch p across, q up; area factor pq

Two-way stretch:

$R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ (anticlockwise; use $-\theta$ for clockwise)

Any angle θ:

$M\begin{pmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{pmatrix} = \begin{pmatrix}x_1' & x_2' & x_3'\\ y_1' & y_2' & y_3'\end{pmatrix}$

All vertices at once:

$\det M = ad - bc$ for $M = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$

Determinant:

$= |\det M| \times (\text{original area})$

Image area:

"P then Q" is the matrix QP — the second transformation sits on the left

Compose:

$\det(QP) = \det Q\ \det P$, so the separate area factors multiply

Area of a composite:

KEY METHODS

- Every 2×2 matrix fixes the origin ($M\mathbf{0} = \mathbf{0}$), so these are rotations, reflections, enlargements or stretches about O. A translation is the one map that is not a matrix multiplication — that is the affine $+b$ term in §9.
- For an enlargement just multiply both coordinates by k. A negative k also sends the point straight through O — a 180° turn as well as a scale.
- An enlargement of factor k scales area by k^2, not k, because it stretches in both directions at once (

EXAM TRAPS TO AVOID

- Multiply the rows of the matrix against the column vector, not the columns. Silently transposing M is the classic slip that produces the wrong image.
- Keep full GDC accuracy for $\cos\theta$ and $\sin\theta$ right to the end, then round once. Rounding θ or the entries early snowballs into a visibly wrong image.
- The origin maps to itself under any 2×2 matrix, so a vertex at O stays put. If the shape is supposed to move away from O, you need a translation (§9), not just a matrix.

$\det = k \cdot k = k^2$).

- For $y = x$ you simply swap the two coordinates; for $y = -x$ swap and negate both. For the axes, only the coordinate across the mirror flips sign.
- Every reflection has $\det = -1$: area is unchanged, but the orientation is reversed (a shape and its mirror image have opposite "handedness").
- Track where $(1, 0)$ lands to check your direction and signs: R_θ sends it to $(\cos\theta, \sin\theta)$, the first column. For 90° that is $(0, 1)$ — straight up, i.e. anticlockwise.
- Put every vertex in one $2 \times n$ matrix and hit multiply once on the GDC, rather than transforming points one at a time — faster and far less error-prone.

- Use $ad - bc$, never $ad + bc$, and take the modulus: a negative determinant still enlarges area — the sign only tells you the orientation has flipped.
- "First P then Q" reads left-to-right in words but is right-to-left in symbols: $C = QP$. Because $QP \neq PQ$ in general, reversing the product gives a different — wrong — image.
- An invariant line's direction is an eigenvector; the stretch along it is that eigenvalue λ. Don't hand back $\det M = \lambda_1 \lambda_2$ (the area factor) as the linear scale factor along a line — they are different numbers.
- A pure translation is not a 2×2 matrix multiplication — it is the $+b$ term. That is precisely why translation is the one map that moves the origin, while every matrix transformation leaves O fixed.