VECTORS — THE WHOLE STRAND ON A PAGE

AA HL 3.12–3.16

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This follows the ten sessions exactly. The **basics** are the 20 things every vectors question is built from — know these cold. The **question types** are the 15 shapes the IB actually sets; each one is a method, not a formula.

Session 1 — Vectors I — notation and algebra AHL 3.12

Basics

Notation. A vector wears two dresses. **Column form** stacks the components; **base-vector form** writes them against $\mathbf{i}, \mathbf{j}, \mathbf{k}$. A zero component means that base vector is simply absent — it is not written.

Examiner's note. Column form is quicker for subtracting position vectors, and a sign slip is easier to spot in it.

Adding and scaling. Scaling multiplies **every** component; adding and subtracting work row by row. There is no cross-talk between rows.

Examiner's note. Scale first and write the two intermediate columns down — most slips here are arithmetic, not method.

AB = b − a. The most-used line in the whole strand. The displacement from A to B is **head minus tail**:

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}.$$

Writing $\mathbf{a} - \mathbf{b}$ gives $\overrightarrow{BA}$ — same length, every sign flipped.

Examiner's note. Say it as you write it: 'to B, from A — so B minus A.'

The midpoint. The midpoint of A and B is the **average of the position vectors**:

$$\mathbf{m} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b}).$$

It is halfway along $\overrightarrow{AB}$, so it is also $\mathbf{a} + \frac{1}{2}\overrightarrow{AB}$ — the same point reached two ways.

Examiner's note. Averaging is the fast route. If a question gives you the midpoint and one end, rearrange: $\mathbf{b} = 2\mathbf{m} - \mathbf{a}$ — the same move as reflecting a point.

Question type: Geometry without coordinates

A figure, two base vectors, and not a single coordinate. How the IB tests whether you understand vectors rather than arithmetic.

1. **The diagonal.** Reading a route through a figure is just addition: $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$. In the parallelogram $\overrightarrow{AC}$ **equals** $\overrightarrow{OB} = \mathbf{b}$ — equal vectors, drawn somewhere else. So the diagonal from the shared corner is the **sum**.
2. **The other diagonal.** One diagonal is the sum; the other is the **difference**: $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$. The same head-minus-tail rule from the roots, now with no coordinates in sight.
3. **A point part-way along.** You have $\overrightarrow{AB}$. Travel a fraction λ of it from A:

   $$\overrightarrow{OP} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}.$$

   The weights always sum to 1 — the signature of a point *on the segment*.
4. **Given as a ratio instead.** A ratio $AP : PB = m : n$ just names $\lambda = \frac{m}{m+n}$, so

   $$\overrightarrow{OP} = \frac{n}{m+n}\mathbf{a} + \frac{m}{m+n}\mathbf{b}.$$

   Note the swap — the weight on $\mathbf{a}$ carries the *far* part of the ratio, because the further you travel from A, the less of $\mathbf{a}$ remains.

Session 2 — Vectors II — magnitude and direction AHL 3.12

Basics

Magnitude. Pythagoras, in as many dimensions as you have:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Never negative, and squaring removes every sign — so $|\overrightarrow{AB}| = |\overrightarrow{BA}|$.

Examiner's note. On Paper 1 leave it as a surd. $\sqrt{29}$ is a finished answer; 5.39 is not.

Unit vector. Dividing by the magnitude strips out the length and keeps only the direction:

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

Check it — the components of a unit vector square-and-sum to exactly 1.

Examiner's note. Write it as one fraction times a column, $\frac{1}{3}\begin{pmatrix}2\\-1\\2\end{pmatrix}$ — the mark scheme's preferred form.

Parallel vectors. Parallel means one is a scalar multiple of the other, $\mathbf{a} = k\mathbf{b}$ — and the **same** k must work in every row. Matching one row and stopping is the commonest error in this sub-topic.

Examiner's note. Get k from a row with no unknown in it, then use it on the row that has one.

Parallel and anti-parallel. $\mathbf{v} = k\mathbf{u}$ makes the two **parallel** whatever k is. The sign of k says which way: $k > 0$ and they point the **same way**; $k < 0$ and they point **opposite ways** — *anti-parallel*, still parallel as lines. If no single k works, they are not parallel at all.

Examiner's note. A direction vector and its negative describe the same line. That is why an answer 'up to scale' accepts $-\mathbf{d}$ as readily as $\mathbf{d}$.

Question type: Three points

Given three points, decide how they sit relative to one another — the standard opening of a Section A vectors question.

1. **The displacement.** Start where every one of these questions starts: $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$. The rest of the question is built on this one column.
2. **Hence the distance.** Distance is the magnitude of the displacement you already have: $AB = |\overrightarrow{AB}|$. Two steps in a fixed order — **subtract, then measure**.
3. **The midpoint.** A midpoint is an **average of position vectors**, not a difference:

   $$\overrightarrow{OM} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b}).$$

   Compare with $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ from part (a). One names a place; the other names a journey.
4. **Show they are collinear.** Three points are collinear when two displacements **from a shared point** are parallel: $\overrightarrow{AC} = k\,\overrightarrow{AB}$. The shared point is what makes it one line rather than two parallel ones — and that k is the ratio.

Session 3 — The scalar (dot) product

AHL 3.13

Basics

The scalar product. Multiply matching components and add:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

The result is a **number**, not a vector — that is what 'scalar' means.

Examiner's note. Keep the signs. A negative dot product is a real answer, and it tells you the angle is obtuse.

What the sign of the dot product tells you. The dot product carries the angle in its **sign**, before you compute anything: $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$, and the magnitudes are never negative. So **positive means acute**, **zero means perpendicular**, **negative means obtuse**. One arithmetic line answers 'is this angle bigger or smaller than 90°'.

Examiner's note. Examiners ask this without the word 'dot product' — 'show the angle is obtuse' means show the dot product is negative, and nothing more.

Angle between vectors. The two forms of the dot product together give the angle:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

Place them **tail to tail**, so $0° \le \theta \le 180°$.

Examiner's note. On Paper 1 stop at the exact cosine; on Paper 2 give the angle to 3 s.f.

Perpendicular. The cleanest condition in the strand:

$$\mathbf{a} \perp \mathbf{b} \iff \mathbf{a} \cdot \mathbf{b} = 0.$$

Set the dot product to zero and solve — almost always one linear equation.

Examiner's note. Substitute your answer back and check the dot product really is zero. Five seconds, one mark.

Session 4 — Geometric vector proofs

AHL 3.12–3.13

Question type: Find the value of k

A configuration with a letter left in it. Each condition — perpendicular, collinear, a given area — turns into one equation in k, and the papers set this more often than any other opening.

1. **Perpendicular.** Write $\overrightarrow{AB}$ with the k still in it — only one component carries it — and $\overrightarrow{AC}$ as usual. Perpendicular means
$$\overrightarrow{AB} \cdot \overrightarrow{AC} = 0,$$
and because k appears once and linearly, that dot product is a **linear equation in** k. One line of algebra.
2. **Collinear.** Collinear means $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are **parallel**: $\overrightarrow{AB} = t\,\overrightarrow{AC}$ for some t. The two components that carry no k fix t straight away; the remaining component then gives k.
3. **Hence the line through all three.** With that k the three points are on one line, so any of them serves as the point and $\overrightarrow{AC}$ as the direction. Nothing new to compute — the parts before it have already produced both pieces.
4. **And with a different k, the area.** Change k and the points stop being collinear, so they now make a genuine triangle. Its area is half the cross product of two sides from the same vertex:
$$\tfrac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|.$$

Question type: Show that it is always true

A Section B proof. No coordinates to grind — the marks are for the argument, and each part is one algebraic step you have to justify rather than compute.

1. **Write the dot product with the angle in it.** Both sides of the identity are about the **angle**, so start by putting the angle into each piece. The dot product's definition does that immediately: $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$, so squaring gives $|\mathbf{u}|^2|\mathbf{v}|^2\cos^2\theta$.
2. **And the cross product with the angle in it.** The cross product carries the **sine** of the same angle: $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$. Squaring gives $|\mathbf{u}|^2|\mathbf{v}|^2\sin^2\theta$ — the sine partner to the piece you just wrote.
3. **Add them and use the Pythagorean identity.** Adding the two gives
$$|\mathbf{u}|^2|\mathbf{v}|^2\left(\sin^2\theta + \cos^2\theta\right),$$
and the bracket is 1. The angle disappears, leaving $|\mathbf{u}|^2|\mathbf{v}|^2$ — which is the result. That single Pythagorean step is the whole proof.
4. **Confirm the right-hand side.** The numbers agreeing does not prove anything on its own — one example never does. But it confirms you have not slipped, and the **argument** in the parts above, written with θ rather than numbers, is what proves it for every pair of vectors.

Session 5 — The vector equation of a line

AHL 3.14

Basics

The forms of a line. A line has three faces and they all say the same thing. **Vector**: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$. **Parametric**: the three rows written out, $x = a_1 + \lambda d_1$ and so on. **Cartesian**: eliminate λ from those rows to get
$$\frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}.$$
Reading a Cartesian form backwards is the same move in reverse: the numbers subtracted from x, y, z are the **point**, the denominators are the **direction**.

Examiner's note. The commonest slip is a sign: $\dfrac{x+4}{3}$ means the point has $x = -4$, not $+4$. Cartesian form subtracts the coordinate.

What the parameter does. In $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, **$\lambda$ is a dial**. Every value gives one point of the line, and every point of the line has exactly one λ. $\lambda = 0$ sits on $\mathbf{a}$; $\lambda = 1$ is one step of $\mathbf{d}$ further on; negative λ goes back the other way.

Examiner's note. To test whether a point lies on a line, solve for λ in the first row — then check that **the same** λ works in the other two. One row agreeing proves nothing.

Question type: The equation of a line

Turn two points into a line, move between its forms, and test whether a point lies on it.

1. **A direction vector.** A line needs **a point on it** and **a direction along it**. Through A and B the direction is $\overrightarrow{AB}$ — the root skill again, doing a new job.

2. **Hence the vector equation.** Point plus direction is the whole equation:

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}.$$

   Either given point works and any parallel direction works — one line has infinitely many correct equations, and the mark scheme accepts all of them.

3. **In Cartesian form.** Cartesian form is the parameter eliminated:

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}.$$

   Numerators give a point, denominators give the direction. Watch the signs — $\frac{x+3}{2}$ means $x_0 = -3$.

4. **Does P lie on L?.** A point lies on the line only if **one single** λ satisfies all three rows. Find λ from one row, then it must check out in the other two. One row agreeing proves nothing.

Session 6 — Relationships between lines

AHL 3.14–3.15

Basics

How two lines can sit. Two lines in space do one of **four** things. **Coincident** — same line, written twice: the directions are multiples *and* a point of one lies on the other. **Parallel and distinct** — directions are multiples, but no shared point. **Intersecting** — not parallel, and one pair of λ, μ satisfies *all three* rows. **Skew** — not parallel, yet no pair works: they pass without meeting, which cannot happen in two dimensions.

Examiner's note. Parallel directions are not enough to call two lines parallel — test whether a point of one lies on the other, or you will call a coincident pair 'parallel'. And skew is a *proved* result, not a shrug: solve two rows, substitute into the third, show it fails.

Question type: Two lines in space

Classify a pair of lines, find where they meet, and measure the angle between them. The densest-examined type in the strand.

1. **How are they related?.** Three questions, in order. **Are the directions parallel?** If yes, the lines are parallel. If no, solve two of the three row-equations for λ and μ, then **substitute into the third**. Holds $\Rightarrow$ intersecting. Fails $\Rightarrow$ skew.

2. **Hence the point of intersection.** The work is already done: the λ that classified the lines is the λ that locates the point. Substitute it into L_1. Substituting μ into L_2 must give the same point — a free check.

3. **And the angle between them.** The angle between two lines is the angle between their **directions** — where they sit is irrelevant:

$$\cos\theta = \frac{|\mathbf{d}_1 \cdot \mathbf{d}_2|}{|\mathbf{d}_1||\mathbf{d}_2|}.$$

   The modulus gives the acute angle: a line has no direction of travel, so $\mathbf{d}$ and $-\mathbf{d}$ describe the same line.

Session 7 — Distance problems with lines

AHL 3.14

Question type: The closest point on a line

A point, a line, and the perpendicular between them. One idea — set a dot product to zero — carries every part, reflection included.

1. **Find the parameter.** The closest point is where $\overrightarrow{PN}$ is **perpendicular to the direction**:

$$\overrightarrow{PN} \cdot \mathbf{d} = 0.$$

   Write N as $\mathbf{a} + \lambda\mathbf{d}$, form $\overrightarrow{PN} = N - \mathbf{p}$, and the condition becomes one equation in one unknown.

2. **Hence the foot of the perpendicular.** Substitute that λ back into the line. This is the three-beat that runs through the whole strand: **find the parameter, substitute back, measure.**

3. **And the shortest distance.** The shortest distance is just $|\overrightarrow{PN}|$ — the magnitude root skill, applied to the displacement you now have. The perpendicular really is shortest: every other point on the line is further away.

4. **And the reflection of P in the line.** N is the **midpoint** of P and its mirror image P' — that is what reflecting in the line means. Rearranging $N = \frac{1}{2}(P + P')$ gives

$$P' = 2N - \mathbf{p}.$$

   One line of work, once you have the foot.

Question type: Two skew lines, and how far apart

The Section B finale. Two lines that never meet still have a closest approach, and the vector perpendicular to both is what measures it.

1. **A direction perpendicular to both.** The shortest route between two skew lines is the one that meets **both** at right angles. A vector along that route is therefore perpendicular to both directions — which is exactly

$$\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2.$$

   Every later part hangs off this one vector.
2. **Where the common perpendicular meets L_1.** Now locate the two ends of that shortest route. Write the two points as $\mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\mathbf{a}_2 + \mu\mathbf{d}_2$; the vector between them must be **parallel to n**, which gives two equations in λ and μ. Solve, and substitute back.
3. **And where it meets L_2.** The other end, the same way. The two feet are the **only** pair of points, one on each line, whose joining vector is perpendicular to both — and the distance between them is the answer to part (b), which is a free check on all of it.
4. **The shortest distance between them.** Nothing new to set up. You located both ends of the shortest route by making the joining vector perpendicular to each direction, so the distance is just the length of the vector between them — the magnitude skill, one more time.

Session 8 — The vector (cross) product, and planes AHL 3.16

Basics

The vector product. Unlike the dot product this returns a **vector**, perpendicular to both inputs:

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}.$$

The middle row is the reversed one — that is the minus sign in the determinant, and where nearly every sign error happens.

Examiner's note. Check it by dotting the answer with both inputs. Both should give exactly zero.

What the cross product means. Two facts, and every cross-product question uses one of them. **Direction**: $\mathbf{u} \times \mathbf{v}$ is perpendicular to *both* — which is where every plane's normal comes from. **Size**: $|\mathbf{u} \times \mathbf{v}|$ is the **area of the parallelogram** they span, so half of it is the area of the triangle.

Examiner's note. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$ — the sine partner to the dot product's cosine. Parallel vectors give zero area, which is another way of testing parallel.

The right-hand rule. The cross product is perpendicular to both — but *which* of the two opposite directions? Point your right hand along **u**, curl your fingers toward **v**, and your **thumb** gives $\mathbf{u} \times \mathbf{v}$. On the axes this cycles:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Examiner's note. Order matters: $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$. Swapping the two vectors flips the normal, which flips the sign of a plane's equation — the same plane, written the other way round.

A plane from two directions. A line needs one direction; a plane needs **two**. Starting from a point **a** and two non-parallel directions **b** and **c** lying in it,

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

sweeps out the whole plane — two dials instead of one. Cross those two directions and you get the normal, which is the bridge to the other two forms.

Examiner's note. The two directions must not be parallel, or they sweep out a line and not a plane. Any two independent directions in the plane will do — three points A, B, C give you $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

The forms of a plane. A plane is pinned down by **one normal and one point on it**. Scalar-product form says every point of the plane has the same projection onto the normal:

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}.$$

Writing $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and multiplying out turns that straight into **Cartesian** form, $ax + by + cz = d$ — the components of **n** *are* the coefficients.

Examiner's note. So you can read a normal straight off any Cartesian equation, and that is usually the first thing a plane question needs.

Question type: A triangle becomes a plane

Three points define a triangle — and the same three points define a plane. One cross product does both jobs.

1. **The cross product.** Both displacements must start from the **same vertex**. Then $\overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to both — and so to the whole plane of the triangle.
2. **Hence the area.** $|\mathbf{u} \times \mathbf{v}|$ is the area of the **parallelogram** spanned by the two vectors, so a triangle is half of it:

$$\text{area} = \tfrac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|.$$

3. **And the plane through them.** The same cross product is the plane's **normal**. Then

$$ax + by + cz = d, \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{n}, \quad d = \mathbf{n} \cdot \mathbf{a}.$$

   This is the cross product's principal job in the course.

Question type: The parallelogram

Three vertices given, the fourth to find, then the diagonals, the area and the plane it sits in. The standard Section B opener, taken from three points to a plane in four steps.

1. **The fourth vertex.** In a parallelogram opposite sides are equal **as vectors**, so $\overrightarrow{AD} = \overrightarrow{BC}$. That rearranges to

$$\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}.$$

   No diagram needed — the vector equation says which vertex is which.
2. **Where the diagonals cross.** The diagonals of a parallelogram **bisect each other**, so they cross at the midpoint of either one. AC is the easier: average **a** and **c** and you are done.
3. **Its area.** Two sides from the same vertex span the shape, so its area is $\left|\overrightarrow{AB} \times \overrightarrow{AD}\right|$ — or equivalently $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$ using the diagonal, since the diagonal and the far side differ by a vector already counted.
4. **And the plane it lies in.** All four vertices lie in one plane, and the cross product from the last part is already its **normal**. Use any vertex for the constant and the Cartesian equation follows in a line.

Session 9 — Intersections and angles with planes

AHL 3.16

Question type: A line meets a plane

Where a line cuts a plane, at what angle, and how far off it a point sits. Contains the sine-versus-cosine distinction the IB tests most years.

1. **Where the line crosses.** Substitute the line's parametric coordinates into the plane equation. It collapses to **one equation in** λ; solve, then put λ back. If the λ terms cancel entirely, the line is parallel to the plane.
2. **At what angle — and why sine.** This one uses **sine**:

$$\sin\theta = \frac{|\mathbf{d} \cdot \mathbf{n}|}{|\mathbf{d}||\mathbf{n}|}.$$

   The formula naturally finds the angle to the **normal**; the angle to the *plane* is its complement, and $\sin^{-1}$ makes that switch for you.
3. **How far Q is from the plane.** No formula. Walk from Q along the normal until you land on the plane: the point $\mathbf{q} + t\mathbf{n}$ lies on Π exactly when

$$\mathbf{n} \cdot (\mathbf{q} + t\mathbf{n}) = d,$$

   which is one linear equation in t. Solve it, and $t\mathbf{n}$ is the step that takes you there — so the distance is $|t|\,|\mathbf{n}|$.
4. **The reflection of Q in the plane.** The same step, taken twice. With the t you just solved, the foot is $F = \mathbf{q} + t\mathbf{n}$; carry on the same distance again and you arrive at the image,

$$Q' = \mathbf{q} + 2t\mathbf{n}.$$

   Equivalently F is the midpoint of Q and Q' — which is how you check it.
5. **And two planes together.** The angle between two planes is the angle between their **normals** — and here it *is* cosine:

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}.$$

   Both objects are already represented by perpendicular vectors, so no complement is needed.

Question type: A line that never meets a plane

Substituting a line into a plane usually gives one value of λ. When the λ terms cancel instead, something has gone right, not wrong — and which of two things it is depends on the constant.

1. **Why lambda disappears.** Substitute the line into the plane and the coefficient of λ is $\mathbf{d} \cdot \mathbf{n}$. If that is **zero**, λ vanishes from the equation entirely and there is nothing left to solve — the line runs along the plane rather than through it.
2. **Parallel, or lying inside it?.** Perpendicular to the normal only says the line is **parallel** to the plane. Two things fit that: the line floats above it, or it lies inside it. Test a single point of the line in the plane's equation — one substitution settles it.

3. **How far off it sits.** Because the line is parallel, every one of its points is the same distance from the plane — so pick the point you were handed and do it the long way. Step from **a** along the normal and ask when you land on the plane:

$$\mathbf{n} \cdot (\mathbf{a} + t\mathbf{n}) = d.$$

One equation, one unknown. The distance is then $|t|\,|\mathbf{n}|$.

4. **And a line that does lie in it.** Change only the point. Same direction, so still parallel — but now the point satisfies the plane's equation, and the line is **contained** in the plane. Every one of its points is a solution, so the line and plane meet in infinitely many places rather than none.

Question type: Three planes together

Two planes meet in a line; a third either pins that line to a point, contains it, or misses it. Reading which of those has happened is the whole skill.

1. **Where two planes meet.** Two non-parallel planes always meet in a **line**. That line lies in both planes, so it is perpendicular to **both** normals — which is exactly what the cross product delivers:

$$\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2.$$

2. **Where all three meet.** Three planes with independent normals meet at exactly **one point**. Solve the three equations together — eliminate one variable between two pairs, then back-substitute. A GDC will do it, but the elimination is what the method marks are for.
3. **Hence the line where two planes meet.** Part (a) gave the **direction** of that line and part (b) gave a **point** that lies on every one of the planes — so it certainly lies on the first two. Direction plus point is a line, and no further solving is needed.
4. **When the third plane contains the line.** Now replace Π_3. If the new normal is a **combination of the first two**, the third plane can no longer pin the line down — it either contains that line or misses it entirely. Which one depends on the constant.
5. **When it misses.** Same normals, different constant. The three planes are still pairwise non-parallel, so each pair still meets in a line — but those three lines are now parallel and distinct. The planes bound a **triangular prism** and nothing lies on all three.

Session 10 — Applications and synthesis

AHL 3.12–3.16

Question type: Two ships

A line equation with time as the parameter. The shape the IB likes to end a Section B question with.

1. **Position at a later time.** Constant velocity is a line with time as the parameter:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

Everything you know about lines transfers — the 'direction vector' is now the velocity.

2. **Its speed. Velocity** is a vector; **speed** is its magnitude — a single positive number. The magnitude root skill, one last time.
3. **When they are closest.** Form the **separation** $\mathbf{s}(t) = \mathbf{r}_A - \mathbf{r}_B$ and minimise its magnitude. Minimising $|\mathbf{s}|^2$ avoids the square root and has its minimum at the same t:

$$t = -\frac{\mathbf{s}_0 \cdot \Delta\mathbf{v}}{|\Delta\mathbf{v}|^2}.$$

Underneath, this is the perpendicularity condition again.

4. **And how close they get.** Put that t back into the separation and take its magnitude. **Find the parameter, substitute back, measure** — the same three-beat as the closest point on a line, and as the line meeting a plane.

Question type: Do they collide?

Two aircraft whose paths cross. Crossing is not colliding — the question is whether they are ever at the crossing point at the same moment, and that is the distinction examiners are testing.

1. **The bearing A is flying on.** A bearing is measured **clockwise from north**, so with x east and y north it is the angle of $\begin{pmatrix} x \\ y \end{pmatrix}$ from the y-axis, not the x-axis:

$$\theta = \arctan\left(\frac{x}{y}\right),$$

adjusted into $0°$–$360°$. The vertical component plays no part — a bearing is a compass direction on the ground.

2. **Which one is faster.** The vector multiplying t is the **velocity**; its magnitude is the **speed**. Direction lives in the vector, size lives in its magnitude — so comparing two aircraft is just comparing two magnitudes.
3. **The angle between their paths.** The angle between two flight paths is the angle between their **velocity** vectors — where the aircraft happen to be does not matter. Taking the modulus of the dot product gives the acute angle, which is what is asked for.
4. **Where the paths cross.** Here is the trap. To find where the **paths** cross you must let each aircraft have its **own** parameter — $\mathbf{r}_A$ at time s and $\mathbf{r}_B$ at time t — because they need not be there together. Using one t for both asks a different question entirely.
5. **So do they collide?.** Two paths crossing says nothing about a collision. A collision needs **the same point at the same time** — one value of t satisfying all three rows simultaneously. Here the two aircraft reach P at different times, so they miss each other.