TOPIC AHL 2.14 · AHL ONLY · WITH AHL 5.15 INTEGRATION · ALGEBRA AND CALCULUS

Partial fractions cheatsheet

Split a messy fraction into simple pieces, find the constants fast, then integrate each piece. Red pen = what I would tell you in class.

cover-up is your fastest friend ↓

AHL only | Paper 1 & Paper 2 | Links to integration & the logistic DE | 5 worked examples

Partial fractions: split one fraction into simple pieces, then every piece is easy

① **Degree check** — top deg ≥ bottom deg? divide first → improper? long division

② **Factorise** — bottom, fully (ax+b) pieces

③ **Write the form** — one term per factor, extra term if repeated → repeated factor = extra term

④ **Find constants** — cover-up · substitute · equate coefficients

⑤ **Use it** — integrate (ln, 1/u) logistic DE → | | inside every ln

add the pieces back to check

1 Proper or improper? Pick the form

check this first

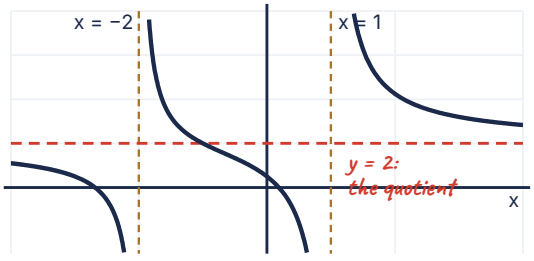
AHL · P1 & P2

You can only split a **proper** fraction: the top has a lower degree than the bottom. If not, divide first. The quotient is a plain polynomial, and only the remainder gets split.

WHICH FORM? ONE TERM PER FACTOR

distinct linear	$\dfrac{f(x)}{(ax+b)(cx+d)} \equiv \dfrac{A}{ax+b} + \dfrac{B}{cx+d}$
repeated linear	$\dfrac{f(x)}{(ax+b)(cx+d)^2} \equiv \dfrac{A}{ax+b} + \dfrac{B}{cx+d} + \dfrac{C}{(cx+d)^2}$
improper	$\dfrac{f(x)}{g(x)} = q(x) + \dfrac{r(x)}{g(x)}$, then split $\dfrac{r(x)}{g(x)}$ as above

$\equiv$ means "equal for every x". Write it, not $=$. Factorise the bottom completely before you start.



$2 + \frac{2}{x-1} + \frac{1}{x+2}$: the polynomial part is the horizontal asymptote.

Trap. Skip the degree check and you set up the wrong form: degree 2 over degree 2 can never equal $\frac{A}{x-1} + \frac{B}{x+2}$ alone.

WORKED EXAMPLE · IMPROPER

Write $\dfrac{2x^2+5x-1}{(x-1)(x+2)}$ in partial fractions.

1. Top and bottom are both degree 2, so it is **improper**. Expand the bottom: $x^2 + x - 2$.
2. Divide: $2x^2 + 5x - 1 = 2(x^2 + x - 2) + (3x + 3)$, so the fraction is $2 + \dfrac{3x+3}{(x-1)(x+2)}$.
3. Split the rest (§2): $3x + 3 \equiv A(x+2) + B(x-1)$. $x = 1$: $A = 2$. $x = -2$: $B = 1$.
4. Answer: $2 + \dfrac{2}{x-1} + \dfrac{1}{x+2}$

check x = 0: left = 1/2, right = 2 − 2 + ½ ✓

2 Finding the constants *the 30-second method*

AHL · P1 & P2

Multiply both sides by the whole bottom to get the **identity line**. It is true for **every** x, so you may substitute any value you like. Pick the values that make brackets vanish.

1. **Degree check.** Improper? Divide first.
2. **Factorise** the bottom fully.
3. **Write the form** (an extra term for a repeated factor).
4. **Identity line:** multiply through by the bottom. Write it down.
5. **Constants:** cover-up, another value, or equate coefficients.
6. **Verify:** add the pieces back, or test one easy x.

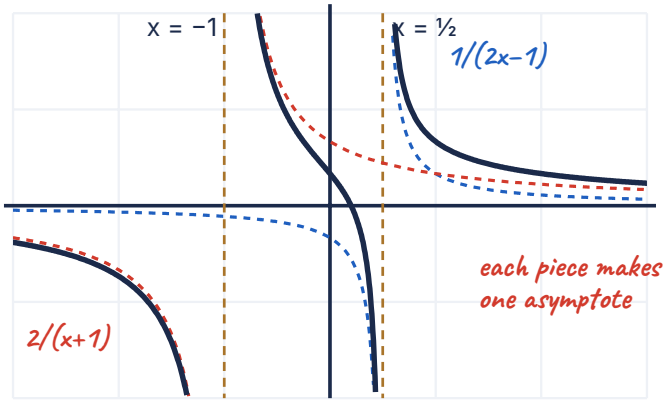
Tool	Use it when
Cover-up: put x = root of the factor	every *distinct* linear factor, and the top power of a repeated one
Another value, e.g. $x = 0$	one constant left after cover-up
Equate coefficients of $x^2, x, 1$	messy roots, or to double-check

WORKED EXAMPLE · COVER-UP WITH A 2X

Split $\dfrac{5x-1}{(2x-1)(x+1)}$.

1. Form $\dfrac{A}{2x-1}+\dfrac{B}{x+1}$. Identity line: $5x-1 \equiv A(x+1)+B(2x-1)$.
2. Root of $2x-1$ is $x=\frac{1}{2}$: $\frac{3}{2}=\frac{3}{2}A$, so $A=1$.
3. Root of $x+1$ is $x=-1$: $-6=-3B$, so $B=2$.
4. Answer: $\dfrac{1}{2x-1}+\dfrac{2}{x+1}$

Check $x=0$: left $=1$, right $=-1+2=1$ ✓



Why cover-up works. At $x=\frac{1}{2}$ the B term vanishes because it is multiplied by $(2x-1)=0$. Only A is left.

Trap. Write the identity line *before* you substitute. A bare "put $x=\frac{1}{2}$" can lose the method mark.

3 Repeated factors *3 factors, 3 constants*

AHL · P1 & P2

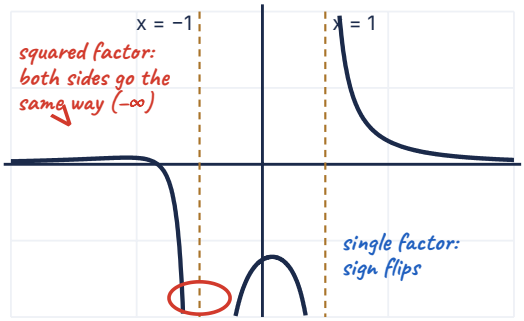
a squared bracket needs two terms

A factor $(x+1)^2$ needs a term for **each power** up to the square: $\frac{B}{x+1}+\frac{C}{(x+1)^2}$. Skip the B term and no choice of constants can work.

FORM FOR A SQUARED FACTOR

$$\frac{f(x)}{(x-1)(x+1)^2} \equiv \frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$A,\ C$	cover-up: $x=1$, then $x=-1$
B	another x, usually $x=0$



Cover-up misses the middle one

It only reaches the *highest* power of a repeated factor, so B needs a second x value. → *x = 0 is usually easiest.*

WORKED EXAMPLE · REPEATED

Split $\dfrac{3x+5}{(x+1)^2(x-1)}$.

1. Form $\dfrac{A}{x-1}+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}$. Identity line: $3x+5 \equiv A(x+1)^2+B(x+1)(x-1)+C(x-1)$.
2. $x=1$: $8=4A$, so $A=2$.
3. $x=-1$: $2=-2C$, so $C=-1$.
4. $x=0$: $5=A-B-C=2-B+1$, so $B=-2$.
5. Answer: $\dfrac{2}{x-1}-\dfrac{2}{x+1}-\dfrac{1}{(x+1)^2}$

check the x^2 coefficients: 0 = A + B = 2 − 2 ✓

4 Integrating the pieces *the payoff*

AHL · P1 & P2

Each piece is now a one-line integral. A $\frac{1}{\text{linear}}$ piece gives a **log**. A $\frac{1}{(\text{linear})^2}$ piece gives a plain fraction, never a log. Always put bars in $\ln|\ |$.

INTEGRALS YOU NEED

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|+C$$

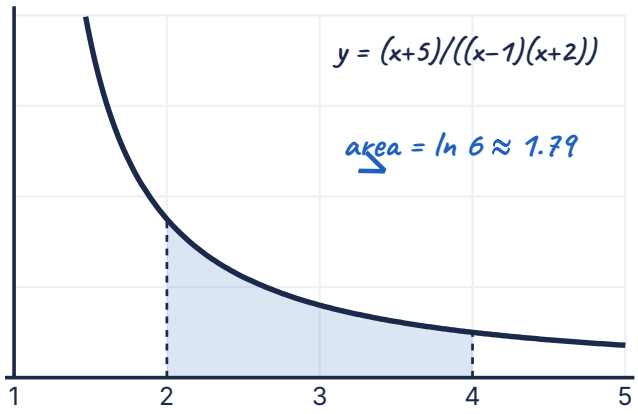
$$\int \frac{1}{(ax+b)^2}dx = \frac{-1}{a(ax+b)}+C$$

$$\int \frac{1}{(ax+b)^n}dx = \frac{(ax+b)^{1-n}}{a(1-n)}+C,\ n\geq 2$$

$$\int \frac{1}{k-x}dx = -\ln|k-x|+C$$

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\frac{x}{a}+C$$

not a partial fraction, but in the booklet



WORKED EXAMPLE · DEFINITE INTEGRAL

Find $\displaystyle\int_2^4 \frac{x+5}{(x-1)(x+2)}dx$**, giving the answer as** $\ln k$.

1. $x+5 \equiv A(x+2)+B(x-1)$. $x=1$: $6=3A$, $A=2$. $x=-2$: $3=-3B$, $B=-1$.
2. $\displaystyle\int_2^4 \left(\frac{2}{x-1}-\frac{1}{x+2}\right)dx = \Big[2\ln|x-1|-\ln|x+2|\Big]_2^4$
3. $= (2\ln 3-\ln 6)-(2\ln 1-\ln 4) = \ln 9-\ln 6+\ln 4$
4. $= \ln\dfrac{9\times 4}{6} = \ln 6 \approx 1.79$

REPEATED FACTOR, INTEGRATED

From §3: $\displaystyle\int\left(\tfrac{2}{x-1}-\tfrac{2}{x+1}-\tfrac{1}{(x+1)^2}\right)dx$

$= 2\ln|x-1|-2\ln|x+1|+\dfrac{1}{x+1}+C$

the last term is not a log!

"Hence". If a question says "hence", use the partial fractions from the earlier part. A different method can lose the marks.

Trap. $\int \frac{1}{k-x}dx = -\ln|k-x|$. The inner function $k-x$ has gradient -1, so the minus sign is the most dropped mark here. Also never forget $+C$ on an indefinite integral.

5 The logistic differential equation *most common application*

AHL · P2 · differential equations

growth that slows down near a limit

When a population P grows towards a limit N, the model is $\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$. Separating the variables leaves $\int \frac{1}{P(N-P)}dP$, and that needs **partial fractions**.

THE KEY IDENTITY (LEARN IT)

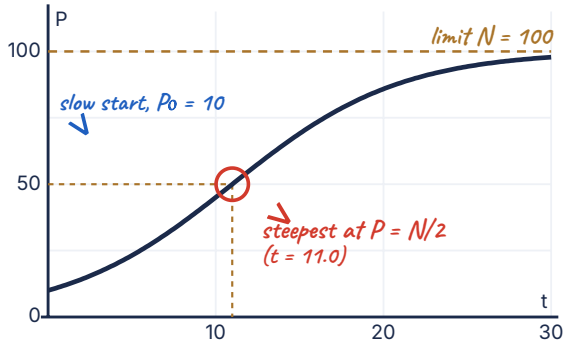
$$\frac{1}{P(N-P)} = \frac{1}{N}\left(\frac{1}{P} + \frac{1}{N-P}\right)$$

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$$
$$\Rightarrow \ln\left|\frac{P}{N-P}\right| = kt + C$$

$$\frac{dP}{dt} = rP(N-P)$$
$$\Rightarrow \ln\left|\frac{P}{N-P}\right| = rNt + C$$

With $P(0) = P_0$:

$$P = \frac{N}{1 + Ae^{-kt}}, \text{ where } A = \frac{N - P_0}{P_0}$$



WORKED EXAMPLE · LOGISTIC

$\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{100}\right)$, $P(0) = 10$. **Find $P(t)$ and when $P = 50$.**

1. Rewrite as $\frac{dP}{dt} = \frac{0.2}{100}P(100 - P)$ and separate: $\int \frac{100}{P(100-P)}dP = \int 0.2\,dt$.
2. Partial fractions: $\frac{100}{P(100-P)} = \frac{1}{P} + \frac{1}{100-P}$.
3. Integrate: $\ln\left|\frac{P}{100-P}\right| = 0.2t + C$.
4. $P(0) = 10$: $C = \ln\frac{10}{90} = -\ln 9$. So $\frac{P}{100-P} = \frac{e^{0.2t}}{9}$.
5. Solve for P: $P = \frac{100}{1 + 9e^{-0.2t}}$
6. $P = 50$ means $e^{0.2t} = 9$, so $t = 5\ln 9 \approx$ **11.0**

fastest growth at P = 50: 0.2 × 50 × ½ = 5 per unit time

Limit check: as $t \to \infty$, $e^{-0.2t} \to 0$, so $P \to 100 = N$.

N or no N?

The N in rNt only appears when the equation is written $rP(N-P)$. In the $kP(1 - \frac{P}{N})$ form it is just kt. → *check which form you were given.*

Before you move on from partial fractions

- ✓ Did I compare the degrees and divide if the top is not smaller?
- ✓ Is the bottom fully factorised before I write the form?
- ✓ Repeated factor: one term for **every** power, so 3 constants for $(x+1)^2(x-1)$?
- Did I write the identity line before substituting?
- ✓ Did I add the pieces back (or test one value of x)?
- ✓ Every log written $\ln|\ |$, and $+C$ on indefinite integrals?
- ✓ $\int \frac{1}{k-x}dx$: minus sign included?
- ✓ "Hence": am I using the earlier partial fractions?