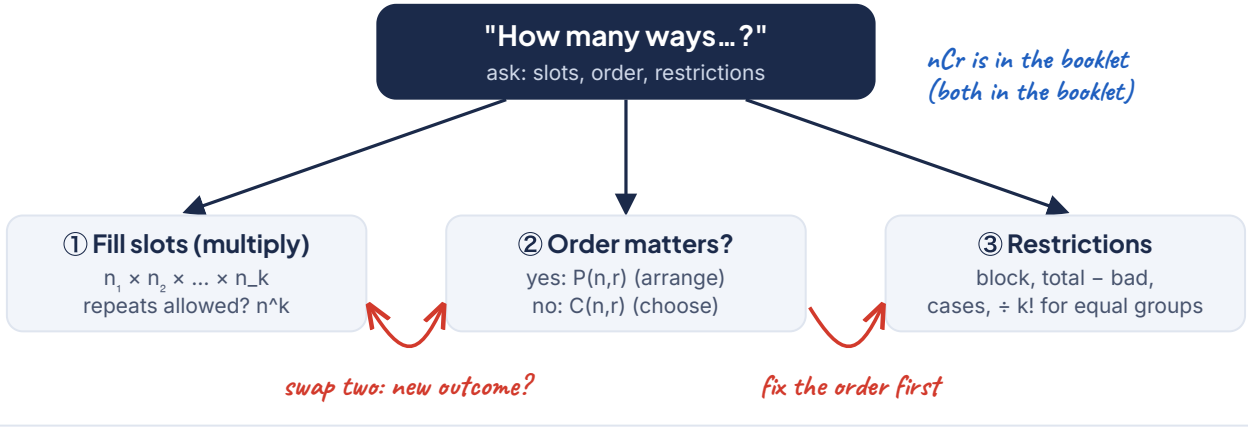
TOPIC SL 1.9 · AHL 1.10–1.11 · THEME NUMBER AND ALGEBRA

Counting principles cheatsheet

Slots, permutations, combinations and the four restriction tricks. Each rule, a picture, and a worked example. Red pen = what I'd tell you in class.

~10 teaching hours | Paper 1 & Paper 2 | nCr, nPr in formula booklet | 7 worked examples

ask "does order matter?" before you calculate ↓



1 Slots, factorials, ${}^nP_r$ and ${}^nC_r$ — the toolkit

SL 1.9 · AHL 1.10 · P1 & P2

count the choices at each stage, then multiply

Most counting is filling slots: how many choices for slot 1, how many for slot 2 once slot 1 is used, and so on. Multiply them. Factorials, ${}^nP_r$ and ${}^nC_r$ are shortcuts for the common cases.

CORE FORMULAS

slots (FCP)	$n_1 \times n_2 \times \cdots \times n_k$	
factorial	$n! = n(n-1)\cdots 2 \cdot 1$	
arrange r of n	${}^nP_r = \dfrac{n!}{(n-r)!}$	booklet
choose r of n	$\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$	booklet
symmetry	$\binom{n}{r} = \binom{n}{n-r}$	
Pascal	$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$	

0! = 1, 1! = 1, 2! = 2, 3! = 6, 4! = 24, 5! = 120, 6! = 720, 7! = 5040, 8! = 40 320. Edges: ${}^nC_0 = 1$, ${}^nC_1 = n$, ${}^nC_n = 1$. And ${}^nP_r = {}^nC_r \times r!$.

repeats allowed: same choices every slot

5 × 5 × 5 = $5^3 = 125$

no repeats: one fewer each slot

5 × 4 × 3 = 60 = P(5,3)

3-digit codes from the digits 1 to 5

WORKED EXAMPLE · CODES

How many 4-digit codes from 0–9 have no repeated digit and do not start with 0?

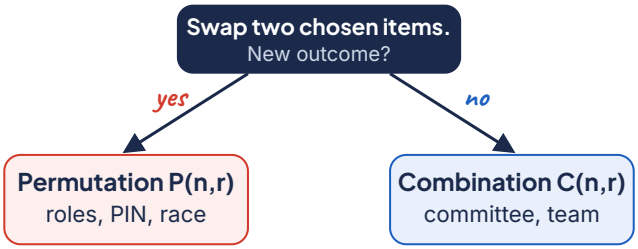
1. Slot 1: not 0, so 9 choices
2. Slot 2: 0 is allowed again but one digit is used: 9
3. Slots 3, 4: 8 then 7. Total 9 × 9 × 8 × 7 = **4536**

Trap: the leading zero. A k-digit number cannot start with 0 (it would have fewer digits). Do slot 1 on its own, then carry on.

2 Permutation or combination? *order or not?*

SL 1.9 · AHL 1.10 · P1 & P2

One test decides it. Swap two of the chosen items. A different outcome means order matters (permutation). The same outcome means it does not (combination).



"increasing order" is secretly a combination: only one way to arrange a chosen set

WORKED EXAMPLE · SAME PEOPLE, TWO QUESTIONS

10 students. (a) Choose a chair, vice-chair and secretary. (b) Choose a team of 3.

1. Roles differ, so order matters: ${}^{10}P_3$ = 10 × 9 × 8 = **720**
2. No roles: ${}^{10}C_3$ = 720 / 3! = **120**
3. Each team of 3 gives 3! = 6 role-assignments: 120 × 6 = 720 ✓

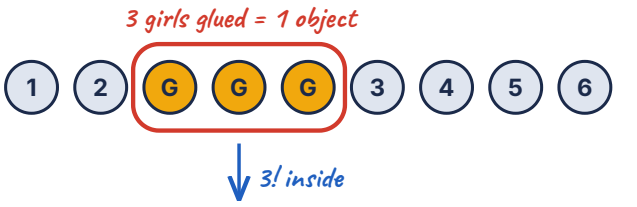
Increasing order. "6 distinct digits from 1–9 in increasing order": choosing the set fixes the order, so ${}^{9}C_6$ = **84**, not ${}^{9}P_6$.

3 Adjacent, not adjacent, "anywhere after" *glue them*

AHL 1.10 · P1 & P2

restriction? count the good, or total minus bad

For "must be together", glue them into one block, arrange the blocks, then arrange inside the block. For "must not be together", count **total − (together)**.



9 − 3 + 1 = 7 objects → 7! outside

7! × 3! = 5040 × 6 = 30 240

RULES

r together, any order inside	$(n-r+1)! \times r!$
fixed order inside ("A just after B")	$(n-1)!$
not together	$n! - (\text{together})$
"A anywhere after B"	$\frac{n!}{2}$

WORKED EXAMPLE · A AND B APART

7 people stand in a line. How many ways if A and B are not next to each other?

1. Total: 7! = 5040
2. A and B together: block of 2, so 6! × 2! = 1440
3. Apart: 5040 − 1440 = **3600**

"A just after B" in this line: 6! = 720 (block fixed)

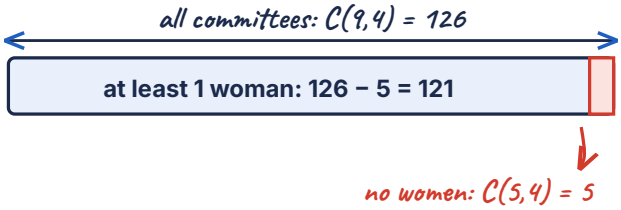
Half by symmetry

"A finishes somewhere after B" in a race of 8: B before A and A before B are equally likely, so 8!/2 = 20 160. → *"immediately after" is the block, "anywhere after" is half.*

4 "At least", "at most" and equal groups *cases or complement*

AHL 1.10 · P1 & P2

"At least 1" is almost always total − none. "At least 2" or more: add up the valid cases (or subtract the invalid ones). For equal teams with no names, divide by $k!$ to remove repeat orderings.



12 people, 3 teams of 4, unnamed teams

C(12,4) × C(8,4) × C(4,4) = 34 650 ordered, ÷ 3! = 5775

WORKED EXAMPLE · AT LEAST 2

A committee of 4 from 5 men and 4 women. How many have at least 2 women?

1. Cases: 2W2M, 3W1M, 4W
2. ${}^4C_2 \times {}^5C_2 + {}^4C_3 \times {}^5C_1 + {}^4C_4 = 60 + 20 + 1 =$ **81**
3. Check by complement: $126 - ({}^5C_4 + {}^4C_1 \times {}^5C_3) = 126 - 5 - 40 = 81$ ✓

EQUAL GROUPS, NO LABELS

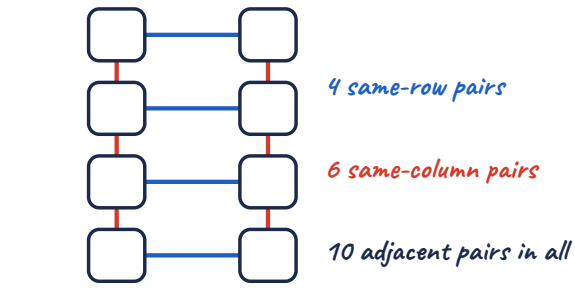
$\frac{n!}{(r!)^k\,k!}$ for k groups of size r. If the groups have names (Red, Blue, Green) do **not** divide by $k!$: $12!/(4!)^3 = 34\,650$.

Trap. "At least 1 of A or B" is not "A in" plus "B in": you double-count when both are in. Subtract the none case from the total instead.

5 Grids, finding n and residue classes *harder patterns*

AHL 1.10 · P1 & P2

Seating grids: count the adjacent pairs first. "Find n": write both counts as expressions in n and solve. Divisibility: sort the numbers by remainder.



pairs = r(c − 1) + c(r − 1)

sum ≡ 0 (mod 3) only when the residues are 0+0+0, 1+1+1, 2+2+2 or 0+1+2

C(6,3) + C(7,3) + C(7,3) + 6·7·7 = 20 + 35 + 35 + 294 = 384

WORKED EXAMPLE · SEATS IN A GRID

8 people sit in a 4 × 2 block of seats. A and B must not be in adjacent seats (row or column).

1. Adjacent pairs of seats: 4(1) + 2(3) = 10
2. Bad: 10 pairs × 2 orders for A, B × 6! for the rest = 14 400
3. Good: 8! − 14 400 = **25 920**

WORKED EXAMPLE · FIND N

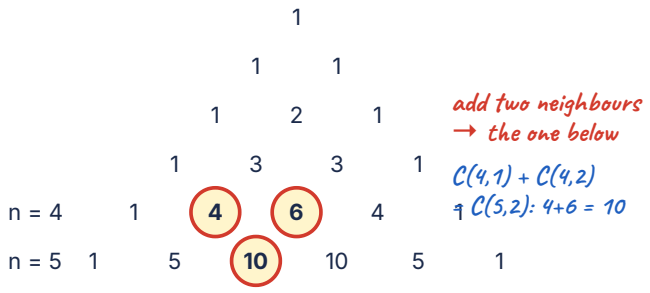
${}^nC_2 + n = 21$. Find n.

1. $n(n-1)/2 + n = 21 \rightarrow n^2 - n + 2n = 42$
2. $n^2 + n - 42 = 0 \rightarrow (n-6)(n+7) = 0$
3. n must be a positive integer, so reject −7: n = **6**

Trap. Read whether "adjacent" means same row only or also above and below. It changes the pair count. When solving for n, always test roots against $n \geq r$ and $n \in \mathbb{Z}^+$.

6 Induction with combinations, and the GDC *Pascal in proofs*

AHL 1.10–1.11 · P1 & P2



WORKED EXAMPLE · PROVE BY INDUCTION

Prove $\Sigma_{r=2}^{n}\ {}^{r}C_{2} = {}^{n+1}C_{3}$ for n ≥ 2.

1. **Base n = 2:** LHS = ${}^{2}C_{2}$ = 1, RHS = ${}^{3}C_{3}$ = 1 ✓
2. **Assume true for n = k** ($k \in \mathbb{Z}^{+}$, k ≥ 2): $\Sigma = {}^{k+1}C_{3}$
3. **n = k + 1:** LHS = ${}^{k+1}C_{3} + {}^{k+1}C_{2} = {}^{k+2}C_{3}$ (Pascal) = RHS ✓
4. **Conclude:** true for n = 2, and true for k ⇒ true for k + 1, so true for all n ≥ 2 by induction.

THE INDUCTION SCRIPT

1. **Base case:** work out both sides as numbers.
2. **"Assume true for n = k"** and write that statement out in full. Not "let n = k".
3. **Step:** write the n = k + 1 left side, use the assumption, apply Pascal.
4. **Conclusion sentence:** the most-dropped mark.

GDC task (Paper 2)	How
${}^{n}C_{r}$, ${}^{n}P_{r}$, n!	MATH → PRB → nCr, nPr, !
solve for n	solve(..., n) or a table

Write the setup first, e.g. ${}^{10}C_{3} \times {}^{7}C_{4}$ = 120 × 35 = 4200, then quote the GDC. Check `solve` roots are integers n ≥ r.

Before you move on from counting

- ✓ Slots: choices at each stage, then multiply. Repeats allowed? n^{k}.
- ✓ Swap two items: new outcome (${}^{n}P_{r}$) or not (${}^{n}C_{r}$)?
- ✓ Number problem: first digit cannot be 0.
- ✓ Together: block × r!. Apart: total − together.
- ✓ "Anywhere after": n!/2. "Immediately after": block, no r!.
- ✓ "At least 1": total − none. "At least 2+": add the cases.
- ✓ Equal unlabelled groups: divide by k!. Named groups: do not.
- ✓ Induction: both sides at base, "assume true for n = k", conclusion sentence.