TOPIC AHL 1.12–1.14 · AHL ONLY · NUMBER AND ALGEBRA

Complex numbers cheatsheet

Cartesian, polar and Euler forms, the Argand diagram, De Moivre and roots, with every step shown. Red pen = what I would tell you in class.

AHL only | Paper 1 & Paper 2 | Formula booklet: polar form, De Moivre | 7 worked examples

think of a point on a plane ↓

1 Cartesian form and the conjugate *the basics*

AHL 1.12 · P1 & P2

i is a number whose square is −1

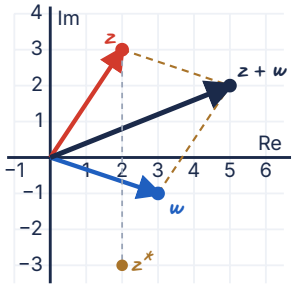
A complex number is $z = a + bi$ with a real part a and an imaginary part b, where $i^2 = -1$. Treat i like any letter in algebra, then replace i^2 by **−1**. The **conjugate** $z^* = a - bi$ is its mirror image in the real axis.

ARITHMETIC

add	$(a+bi)+(c+di) = (a+c)+(b+d)i$
multiply	$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$
divide	$\dfrac{a+bi}{c+di} = \dfrac{(a+bi)(c-di)}{c^2+d^2}$

CONJUGATE RULES

$zz^* = \lvert z\rvert^2 = a^2+b^2$	always real
$z + z^* = 2a$	real
$z - z^* = 2bi$	imaginary
$(zw)^* = z^*w^*,\ (z/w)^* = z^*/w^*,\ (z^n)^* = (z^*)^n$	



z = 2 + 3i
w = 3 − i
z + w = 5 + 2i
add like vectors: complete the parallelogram
z* = 2 − 3i is the mirror in the Re axis

WORKED EXAMPLE · DIVISION

Write $\dfrac{2+3i}{1-2i}$ in the form $a+bi$.

1. Multiply top and bottom by the conjugate of the bottom, $1+2i$.
2. Top: $(2+3i)(1+2i) = 2+4i+3i+6i^2 = -4+7i$.
3. Bottom: $(1-2i)(1+2i) = 1+4 = 5$.
4. Answer: $-\dfrac{4}{5}+\dfrac{7}{5}i$

Trap. $i^2 = -1$, not $+1$. In a product the term $bi \times di$ is $-bd$. Forgetting that sign is the most common lost mark in this section.

Prove a number is real or imaginary. Show $X = X^*$ (real) or $X = -X^*$ (purely imaginary). Equating two complex numbers? Match real parts and imaginary parts separately.

2 Modulus, argument and the Argand diagram *draw the point*

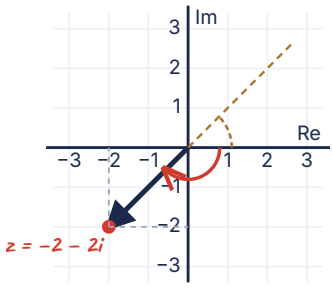
AHL 1.12–1.13 · P1 & P2

Plot $z = a + bi$ as the point (a, b). Its distance from the origin is the **modulus** $|z|$, and the angle from the positive real axis is the **argument** $\arg z$, taken in $(-\pi, \pi]$.

MODULUS AND ARGUMENT RULES

$\lvert z\rvert = \sqrt{a^2 + b^2}$	$\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$
$\lvert z_1/z_2\rvert = \dfrac{\lvert z_1\rvert}{\lvert z_2\rvert}$	$\lvert z^n\rvert = \lvert z\rvert^n$
$\arg(z_1 z_2) = \arg z_1 + \arg z_2$	$\arg(z^n) = n \arg z$
$\arg(z_1/z_2) = \arg z_1 - \arg z_2$	$\arg z^* = -\arg z$
$\lvert z^*\rvert = \lvert -z\rvert = \lvert z\rvert$	$\arg(-z) = \arg z \pm \pi$

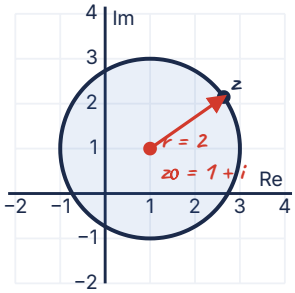
Arguments are only defined up to 2π: add or subtract 2π to bring the answer back into $(-\pi, \pi]$. Triangle inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$.



$|z| = \sqrt{(4 + 4)}$
$= 2\sqrt{2} \approx 2.83$

$\arctan(-2/-2) = \pi/4$
points into Q1: wrong!

z is in Q3, so
$\arg z = \pi/4 - \pi$
$= -3\pi/4$



$|z - z_0| = 2$
a circle: centre z_0, radius 2

$|z - z_0| < 2$
is the inside

$|z_1 - z_2|$ = distance between the points

Trap. $\arctan(b/a)$ only gives the right argument in quadrants 1 and 4. Sketch the point first, then add or subtract π for quadrants 2 and 3.

3 Polar and Euler forms *multiply with angles*

AHL 1.13 · P1 & P2

The same point can be written with its distance and direction. Multiplying in Cartesian form is slow. In polar or Euler form you just multiply the moduli and **add the arguments**.

THREE FORMS OF A COMPLEX NUMBER

IN BOOKLET

Cartesian	$z = a + bi$
polar	$z = r(\cos\theta + i\sin\theta) = r\,\mathrm{cis}\,\theta$
Euler	$z = re^{i\theta}$
polar → Cartesian	$a = r\cos\theta, \quad b = r\sin\theta$
Cartesian → polar	$r = \sqrt{a^2 + b^2}, \quad \theta = \arg z$

PRODUCTS AND QUOTIENTS

$$r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2\, e^{i(\theta_1 + \theta_2)}$$

$$\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2}\, e^{i(\theta_1 - \theta_2)}$$

$$(re^{i\theta})^* = re^{-i\theta}, \quad -re^{i\theta} = re^{i(\theta \pm \pi)}$$

Euler's identity $e^{i\pi} + 1 = 0$ *is the case* $r = 1,\ \theta = \pi$

WORKED EXAMPLE · MULTIPLY IN POLAR FORM

Find $(1 + i)(1 - \sqrt{3}\,i)$ using Euler form, then check it.

1. $1 + i$: $r = \sqrt{2}$, $\theta = \frac{\pi}{4}$. So $1 + i = \sqrt{2}\,e^{i\pi/4}$.
2. $1 - \sqrt{3}\,i$: $r = 2$, in Q4, $\theta = -\frac{\pi}{3}$. So $= 2e^{-i\pi/3}$.
3. Multiply: $2\sqrt{2}\,e^{i(\pi/4 - \pi/3)} = 2\sqrt{2}\,e^{-i\pi/12}$
4. As $a + bi$: $2\sqrt{2}(\cos 15^\circ - i\sin 15^\circ) \approx 2.73 - 0.73i$.
5. Check by expanding: $1 - \sqrt{3}i + i + \sqrt{3} = (1 + \sqrt{3}) + (1 - \sqrt{3})i \approx 2.73 - 0.73i$ ✓

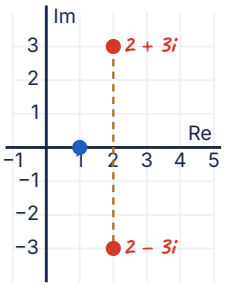
Trap. $\mathrm{cis}\,\theta$ is shorthand for $\cos\theta + i\sin\theta$. It is not $\cos\theta \times i\sin\theta$, so never split it as a product.

Habit. Convert to Euler form before you multiply, divide or raise to a power. Add and subtract in Cartesian form.

4 Polynomials and the conjugate root theorem *roots come in pairs*

AHL 1.14 · P1 & P2

If a polynomial has **real** coefficients and α is a root, then α^* is a root too. The pair multiplies to a real quadratic: $(z-\alpha)(z-\alpha^*) = z^2 - 2\,\mathrm{Re}(\alpha)\,z + |\alpha|^2$.



complex roots of a real polynomial come in mirror pairs

a real root sits on the Re axis

sum of roots = 5 = −(coeff of z^2)

Roots of	Sum	Product
$az^2 + bz + c$	$-\frac{b}{a}$	$\frac{c}{a}$
$az^3 + bz^2 + cz + d$	$-\frac{b}{a}$	$-\frac{d}{a}$

For the cubic, the sum of the products of pairs of roots is $\frac{c}{a}$.

WORKED EXAMPLE · CUBIC WITH A COMPLEX ROOT

$p(z) = z^3 - 5z^2 + az - 13$ has real a and a root $2+3i$. Find a and all roots.

1. Real coefficients, so $2-3i$ is also a root.
2. Sum of roots $= 5$: third root $= 5 - (2+3i) - (2-3i) = 1$.
3. Check the product: $1 \times (2+3i)(2-3i) = 13$ ✓ (should be $-\frac{-13}{1} = 13$).
4. Pairs: $1(2+3i) + 1(2-3i) + 13 = 17$, so $a =$ **17**
5. Roots: $1,\ 2 \pm 3i$

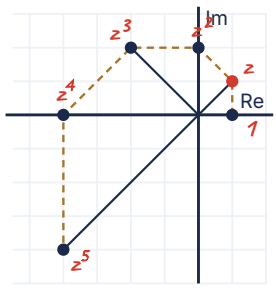
Trap. The theorem needs **all coefficients real**. If one coefficient is complex, α^* is not automatically a root. Also, "sum $= -\frac{b}{a}$, product $= \frac{c}{a}$" is for quadratics only. The product of cubic roots is $-\frac{d}{a}$.

5 De Moivre's theorem *powers = turn and scale*

AHL 1.14 · P1 & P2

raising to a power turns the point and scales it

For any integer n, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. With a modulus, $\left(re^{i\theta}\right)^n = r^n\,e^{in\theta}$: the **modulus** is raised to the power and the angle is multiplied by n.



z = 1 + i = √2 e^(iπ/4)

each power: × √2 in size, turn 45° more

z^4 = −4: modulus $(\sqrt{2})^4 = 4$, angle $4 \times 45° = 180°$

DE MOIVRE *IN BOOKLET*

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$

with $z = e^{i\theta}$: $z^n + z^{-n} = 2\cos n\theta$

$z^n - z^{-n} = 2i\sin n\theta$

WORKED EXAMPLE · A POWER

Write $(\sqrt{3}+i)^5$ in the form $a+bi$.

1. $r = \sqrt{3+1} = 2$, $\theta = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$ (Q1).
2. $(\sqrt{3}+i)^5 = 2^5 e^{i5\pi/6} = 32\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$
3. $= 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) =$ $-16\sqrt{3} + 16i$

WORKED EXAMPLE · A FOURTH POWER

Write $\cos^4\theta$ in terms of multiple angles.

1. With $z = e^{i\theta}$: $(2\cos\theta)^4 = (z + z^{-1})^4$.
2. Expand: $z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4} = (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$.
3. $= 2\cos 4\theta + 8\cos 2\theta + 6$, so $16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$.
4. $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

TRIG FROM DE MOIVRE

Expand $(\cos\theta + i\sin\theta)^3$ with the binomial theorem, then match real and imaginary parts to $\cos 3\theta + i\sin 3\theta$:

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$, $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$

Trap. De Moivre needs the base written as $\cos\theta + i\sin\theta$ (modulus 1), or as $re^{i\theta}$. For n a fraction you get only one of the n values, so use the roots formula in §6.

Induction proof (examinable), $n \in \mathbb{Z}^+$: base $n = 1$ is trivial. Assume true for k. Then $(\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$, and $\cos(A+B)$, $\sin(A+B)$ turn it into $\cos(k+1)\theta + i\sin(k+1)\theta$.

6 The nth roots of a complex number

count them: exactly n

AHL 1.14 · P1 & P2

$z^n = w$ has exactly **n** solutions. They all have the same modulus and sit equally spaced on a circle, so they form a regular n-sided polygon centred at the origin.

THE NTH ROOTS FORMULA

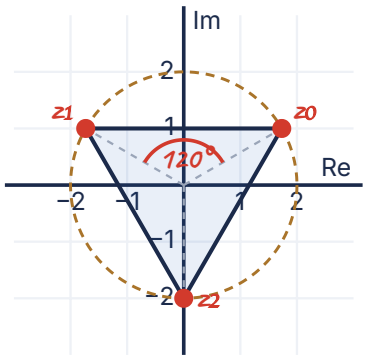
$$z_k = R^{1/n}\, e^{i(\phi + 2\pi k)/n},\quad k = 0, 1, \ldots, n-1$$

modulus $R^{1/n}$ for every root; angle step $\dfrac{2\pi}{n}$

sum of the roots $= 0$; product $= (-1)^{n+1}w$

roots of unity: $z^n = 1$, $z_k = e^{2\pi i k/n}$

With $\omega = e^{2\pi i/n}$: $1 + \omega + \cdots + \omega^{n-1} = 0$ and $\omega^k + \omega^{n-k} = 2\cos\frac{2\pi k}{n}$.



$z^3 = 8i$

3 roots, radius 2

regular triangle

$z_0 = \sqrt{3} + i$

$z_1 = -\sqrt{3} + i$

$z_2 = -2i$

they add up to 0

Trap. A quartic has 4 roots, never 2 or 3. Run k all the way to $n-1$, and bring each argument back into $(-\pi, \pi]$.

WORKED EXAMPLE · CUBE ROOTS

Solve $z^3 = 8i$.

1. Euler form: $8i = 8e^{i\pi/2}$, so $R = 8$, $\phi = \frac{\pi}{2}$, $n = 3$.
2. $z_k = 8^{1/3}e^{i(\pi/2 + 2\pi k)/3} = 2\,e^{i(\pi/6 + 2\pi k/3)}$ for $k = 0, 1, 2$.
3. $z_0 = 2e^{i\pi/6} = \sqrt{3} + i$, $\quad z_1 = 2e^{i5\pi/6} = -\sqrt{3} + i$, $\quad z_2 = 2e^{i3\pi/2} = -2i$ (or $2e^{-i\pi/2}$).
4. Check: $(\sqrt{3} + i)^3 = 8e^{i\pi/2} = 8i$ ✓. **Three roots, forming an equilateral triangle.**

7 The C + iS series trick *one geometric series*

AHL 1.14 · P1 & P2

To sum $\sum r^n \cos n\theta$, build $C + iS = \sum r^n e^{in\theta}$. That is a **geometric series** with ratio $re^{i\theta}$, so one formula does both sums at once.

THE SERIES FORMULAS

For $|r| < 1$: $C = \sum_{n=0}^{\infty} r^n \cos n\theta$, $S = \sum_{n=0}^{\infty} r^n \sin n\theta$

$$C + iS = \frac{1}{1 - re^{i\theta}} = \frac{1 - re^{-i\theta}}{1 - 2r\cos\theta + r^2}$$

So $C = \dfrac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2}$, $S = \dfrac{r\sin\theta}{1 - 2r\cos\theta + r^2}$.

multiply by the conjugate of the denominator, not the denominator

WORKED EXAMPLE · A SERIES

Find $1 + \frac{1}{2}\cos\frac{\pi}{3} + \frac{1}{4}\cos\frac{2\pi}{3} + \ldots$ (to infinity).

1. Here $r = \frac{1}{2}$, $\theta = \frac{\pi}{3}$, $\cos\theta = \frac{1}{2}$. Sum $= C$.
2. Denominator: $1 - 2(\frac{1}{2})(\frac{1}{2}) + \frac{1}{4} = \frac{3}{4}$. Numerator: $1 - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$.
3. $C = 1$ (and $S = \dfrac{\frac{1}{2} \cdot \frac{\sqrt{3}}{2}}{3/4} = \dfrac{1}{\sqrt{3}}$)

Before you move on from complex numbers

- ✓ Did I use $i^2 = -1$ in every product?
- ✓ Division: multiplied by the conjugate of the **bottom**?
- ✓ Argument: sketched the point and fixed the quadrant?
- ✓ Argument inside $(-\pi, \pi]$ after adding or subtracting?
- ✓ Switched to Euler form for multiplying, dividing and powers?
- ✓ Polynomial: real coefficients, so the conjugate root is written down?
- ✓ $z^n = w$: all n roots found, $k = 0$ to $n - 1$?
- ✓ De Moivre base has modulus 1, or I pulled out r^n?